Clunk versus Twang: Collisions of Spheres, Cylinders, and Models with Internal Degrees of Freedom

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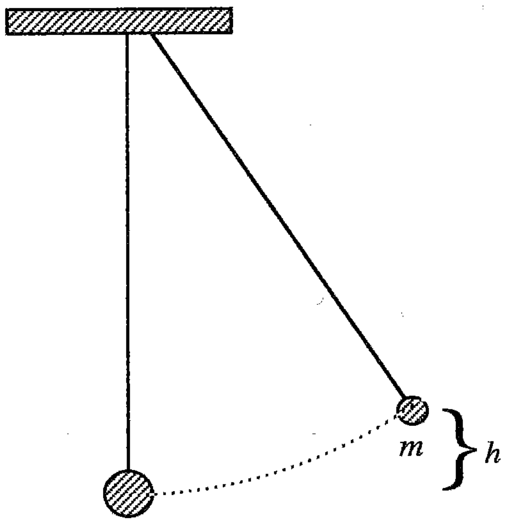
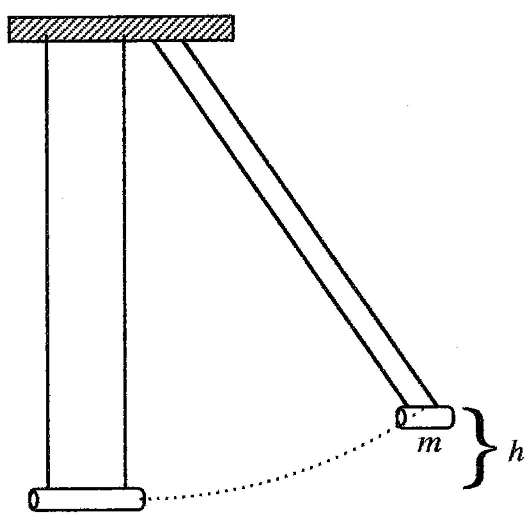
Abstract

The collision of two hard spheres made from the same material is elastic when the velocities are low enough to prevent permanent bending. The longitudinal (end-to-end) collision of hard cylinders, however, is not always elastic. In this thesis, two models are used to explore the collision of spheres and the collision of cylinders. The ratio of the outgoing to the incoming speeds of colliding cylinders constructed from the same material and with identical diameters is found to be the ratio of the shorter cylinder's length to the length of the longer cylinder. No adequate explanation of the sphere's behavior is found with either model.

Introduction: What You Don't Know Might Hurt You



[I.1] -- Two spherical pendulums are strung next to each other. One bob is released at a higher position.



[I.2] -- Two cylindrical pendulums are strung next to each other. One bob is released at a higher position.

Suppose that two pendulums are strung next to each other so that the pendulum bobs are barely touching when they are at rest. Both bobs are steel spheres, and one of the bobs has a smaller radius so that it has half of the mass of the other bob. The strings are massless. The smaller bob is moved upwards to a higher position along the path of its swing, and it is released.

What heights will the bobs reach after the collision? Most physics students and physicists will sit down and calculate that the small bob will rise to one-ninth of its original height and the large bob will rise to a height four-ninths as high as the small bob's original height. To make this prediction, they presume that momentum is conserved during the bobs' collision, and that kinetic energy is also conserved because the bobs are made of steel. Their results are quite accurate.

Now suppose that the same pendulums are set up, except that this time the bobs are steel cylinders with equal diameters and different lengths. Two strings are attached to each cylinder so that they will collide end to end. Again, what heights will the bobs reach after the collision? Most physicists and physics students would again assume that momentum and kinetic energy are conserved and calculate that the small bob will rise to one-ninth of its original height and the large bob will rise to a height four-ninths as high as the small bob's original height. This time their results are wrong.

According to Auerbach,[1] what happens is that the small cylinder ends up motionless at the bottom of the pendulum swing while the big cylinder ends up at a height one-fourth as high as the small cylinder's initial height. The answer is different because kinetic energy is not conserved.

In this thesis, I attempt to explain the differences between the collisions of spheres and cylinders.

Auerbach gave a dynamical explanation of the collision of cylinders that predicts the behavior I have described. By his method, inelasticity is to be expected because some kinetic energy is lost to vibrations that form in the rods during the collision. This is not surprising, however, because we expect that banging on an object excites it, causing it to vibrate. The kinetic energy lost to vibrations ought to be considered when explaining what happens during a collision.

In this light, it is astonishing that the collision of hard spheres is essentially elastic. No vibrations of significant energy seem to be produced in the spheres. Some energy is lost in deforming of the material whence the spheres are made, and creating noise and other aerodynamic phenomena, but there is no equivalently large loss observed in the collision of spheres as the fifty percent loss that occurs in the cylinder collision just described.

The traditional treatment of collisions avoids the complexities of the interactions occurring in a collision by finding simple relations between the initial and final conditions. The prime difficulties introduced by this approach are errors of omission. Deciding what can be ignored and what needs to remain to get the desired results is not always trivial. It is hoped that necessary and sufficient rules for solution are attainable without explicit reference to the equations of motion, but if these rules cannot be found easily, then a more careful investigation must be made. The analysis of the vibrations formed during collisions requires precisely this kind of investigation. There is, however, a way to quantify most of the behavior during collision.

As an analogy, suppose we were measuring the amount of water in a glass, but we spilled some while measuring. It would be very difficult to determine the original amount of water unless we were very lucky and perhaps happened to catch all the spilt water in another container. If this were the case, we could just measure the amount of water in each container and add them up. On the other hand, perhaps we know

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[1]D. Auerbach, "Colliding Rods: Dynamics and Relevance to Colliding Balls," American Journal of Physics **62** (6) (June 1994): 522-525.

something about the way the water spilled, e.g., exactly one half of the water was spilled. In this case we just measure the remainder and multiply by two. This second type of measurement is similar to how we shall treat collisions. Rather than giving all the specifics of the vibrations formed by the collisions, we will present a single number that will cover most of the information we need to know.

Usually, the **coefficient of restitution** is used to give the results of a collision. It is defined for two-body collisions as the ratio of the relative final speed to the relative initial speed. Thus, if the coefficient is one, the speed of approach is equal to the speed of recession, and the collision is said to be elastic. If the coefficient is greater than one, the speed of recession is greater than the speed of approach. If the coefficient is less than one, the speed of approach is greater. Knowing the coefficient of restitution in a particular collision is like knowing that half of the water was spilt, it allows us to use conservation of kinetic energy even when we cannot measure all the energy.

I should emphasize that "coefficient" is a poor name. It would be better to call the coefficient of restitution the "function of restitution", to emphasize that it is a complicated function of many variables. It depends on many variables because it must account for all the hidden sources and sinks of energy. What you don't know can hurt you in collision mechanics: there may be unaccounted-for variables hidden in the coefficient of restitution that greatly affect the velocities. Some quantities that have been found to affect the coefficient of restitution are the momenta of the colliding objects,[2] the density of the objects,[3] and the ambient atmospheric density.[4] For convenience, I will call the variables that can be measured and accounted for **external**, and those that are hidden in the coefficient of restitution I will call **hidden** or **internal**.

The goal of this thesis is to theoretically determine the coefficient of restitution for the collision of spheres and for the collision of cylinders. The collisions occur in vacuum and the colliding objects are not permanently bent during the collision. They possess energy in two forms, internal vibrational energy and energy from the motion of the object as a whole.

In chapter one, some techniques of collision mechanics are presented. In chapter two and chapter three, two models of objects with internal degrees of freedom are presented to try to understand the collision of cylinders and the collision of spheres. Chapter two

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[2]J. P. Andrews, "Impact of Spheres of Soft Metals," London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science **53** (1929): 781-800, Phil. Mag. **58** (1930): 593-610.

[3]D. B. Deodhar, "On the Collision of Spherical Bodies of Unequal Diameters and Densities at Very Low Velocities," Phil. Mag. **48** (1924): 89-96.

[4]S. Banerji, "On Aerial Waves Generated by Impact," Phil. Mag. **35** (1918): 96-111.

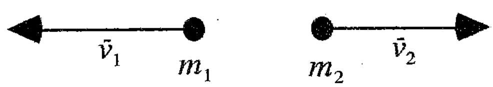
studies collisions of chains of masses tied together by strings. Chapter three is a discussion of collisions of chains of masses tied together by springs.

Chapter One:
The Basics

In this first chapter, I review portions of traditional collision theory by way of solving three examples. The first example is the collision of two point particles, the paradigm for all collisions in the traditional theory. In the second example, the particles are allowed to exchange mass. This change causes the collision to be inelastic. The final example is a collision of hockey pucks that involves rotational kinetic energy. This problem has a coefficient of restitution that is not a constant. The basic conservation laws, the coefficient of restitution, the center of mass, the relative velocity, and the eigenspace basis are all presented while solving the examples.

The first problem begins with two point particles approaching each other. The particles are constrained to one dimension and are not allowed to change mass, vibrate, or otherwise deform internally. The particles are free of external forces.

[1.1] -- The point particles are moving towards each other before the collision.



[1.2] -- The point particles are moving apart after the collision.[1]

Later, the particles begin to interact with a repulsive force (or forces). This force becomes infinitely strong as the particles approach the same location, so the particles cannot pass each other: the particle on the left remains on the left, and the particle on the right remains on the right. They interact for a while, and then they cease their interaction.

We want to find the velocities of the particles after the interactions stop, given their initial velocities.

It can be shown easily that momentum is always conserved when Newton's third law is followed during a collision. More generally, Noether's theorem proves that the conservation of momentum is due to the uniformity of space.[2]

Thus, we begin with the conservation of momentum.

$$m_1 v_1 + m_2 v_2 = m_1 \bar{v}_1 + m_2 \bar{v}_2 \qquad [1.1]$$

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[1]Throughout, a bar (¯) shall be used to signify the value of a quantity after the collision is complete.

[2]H. Goldstein. *Classical Mechanics,* 2nd ed. (Reading, Menlo Park, London, Don Mills, Sydney: Addison-Wesley Publishing Co. 1980) pp.588-600; D. Griffiths. *Introduction to Elementary Particles.* (New York, Chichester, Brisbane, Toronto, Singapore: John Wiley & Sons, 1987), pp. 103, 105, 117, 370.

The conservation of total energy, like conservation of momentum, involves Noether's theorem, and so is generally accepted. In the case of the point particles, we decided beforehand that the particles are not allowed to change mass or deform internally, so all of their energy must be kinetic. If there is no other form of energy where some of the total energy might hide, conservation of kinetic energy is a consequence of the conservation of total energy.

$$\frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 = \frac{1}{2}m_1 \bar{v}_1^2 + \frac{1}{2}m_2 \bar{v}_2^2 \qquad [1.2]$$

We are not interested in the positions of the objects, and before and after the collision the velocities are constant, so the higher derivatives of position are zero. Thus only first derivatives (velocities) concern us. The conservation of kinetic energy and momentum are quadratic and linear in the velocities, respectively. These conservation laws provide two independent constraints on the two final velocities of the particles: all the constraints necessary to specify the final velocities. All that remains is to solve for these velocities. Equation [1.3] is another form of the conservation of energy.

$$\frac{1}{2}m_1 \bar{v}_1^2 - \frac{1}{2}m_1 v_1^2 = -\left(\frac{1}{2}m_2 \bar{v}_2^2 - \frac{1}{2}m_2 v_2^2\right) \qquad [1.3]$$

The factorization of equation [1.3] is

$$m_1\left(\bar{v}_1 - v_1\right)\left(\bar{v}_1 + v_1\right) = -m_2\left(\bar{v}_2 - v_2\right)\left(\bar{v}_2 + v_2\right) \qquad [1.4]$$

The conservation of momentum may be written similarly.

$$m_1\left(\bar{v}_1 - v_1\right) = -m_2\left(\bar{v}_2 - v_2\right) \qquad [1.5]$$

Division of equation [1.4] by equation [1.5] yields the relationship between the difference of the velocities before the collision and the difference of the velocities after the collision.

$$v_2 - v_1 = -\left(\bar{v}_2 - \bar{v}_1\right) \qquad [1.6]$$

Equation [1.6] proves that the speed of approach is equal to the speed of recession. When, as in this collision, the incoming speed is equal to the outgoing speed, we say that the collision is **elastic**.

The only conditions used to show that the point particle collision is elastic were the conservation of kinetic energy, the conservation of momentum, and the conservation of the particle masses. Any two-body collision is also elastic under these conditions.

We now must find the final velocities. The sum of the velocities weighted by the masses[3] is shown to be conserved by the conservation of momentum (equation [1.1]).

[3]No pun intended.

This condition and equation [1.6] are two linear, independent constraints on the velocities, and they can be used in linear combination to produce the velocities. A little algebra produces equations [1.7] and [1.8], the solution to the point particle collision.

$$\begin{aligned}\bar{v}_1 &= \frac{m_1\bar{v}_1 + m_2\bar{v}_2 + m_2\bar{v}_1 - m_2\bar{v}_2}{M} \\ &= \frac{m_1\bar{v}_1 + m_2\bar{v}_2}{M} - \frac{m_2}{M}\left(\bar{v}_2 - \bar{v}_1\right) = \frac{m_1 v_1 + m_2 v_2}{M} + \frac{m_2}{M}\left(v_2 - v_1\right) \\ &= \frac{\left(m_1 - m_2\right)v_1 + 2m_2 v_2}{M}\end{aligned} \qquad [1.7]$$

And

$$\begin{aligned}\bar{v}_2 &= \frac{m_1\bar{v}_1 + m_2\bar{v}_2 - m_1\bar{v}_1 + m_1\bar{v}_2}{M} \\ &= \frac{m_1\bar{v}_1 + m_2\bar{v}_2}{M} + \frac{m_1}{M}\left(\bar{v}_2 - \bar{v}_1\right) = \frac{m_1 v_1 + m_2 v_2}{M} - \frac{m_1}{M}\left(v_2 - v_1\right) \\ &= \frac{2m_1 v_1 + \left(m_2 - m_1\right)v_2}{M}\end{aligned} \qquad [1.8]$$

Equations [1.7] and [1.8] use the **total mass**, defined in equation [1.9].

$$M \equiv m_1 + m_2 \qquad [1.9]$$

Now that we have found the solution to the point particle problem, I would like to make a few notational simplifications before we go on to the second example.

We can represent the velocities with an **initial velocity vector**, $\mathbb{v}$, and a **final velocity vector**, $\bar{\mathbb{v}}$. The masses are formed into a **mass matrix**, $\mathbb{M}$.[4]

$$\mathbb{v} \equiv \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \; \bar{\mathbb{v}} \equiv \begin{pmatrix} \bar{v}_1 \\ \bar{v}_2 \end{pmatrix}, \; \mathbb{M} \equiv \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \qquad [1.10]$$

The solution is expressed as a **collision matrix**, $\mathbb{C}$, that gives the relationship between the initial velocity vector and the final velocity vector. It is helpful to think of applying the collision matrix as "colliding the velocities."

$$\bar{\mathbb{v}} = \mathbb{C}\,\mathbb{v} \qquad [1.11]$$

The collision matrix for the point particles is easily gleaned from equations [1.7] and [1.8].

$$\mathbb{C} = \frac{1}{M}\begin{pmatrix} m_1 - m_2 & 2\,m_2 \\ 2\,m_1 & m_2 - m_1 \end{pmatrix} \qquad [1.12]$$

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[4]Characters in outline (e.g., $\mathbb{M}$) represent matrices or vectors.

Of course, the conservation laws can also be represented in this notation. The conservation of energy is given by equation [1.13].[5]

$$\frac{1}{2}\,\mathbb{v}^{\lrcorner}\,\mathbb{M}\,\mathbb{v} = \frac{1}{2}\,\bar{\mathbb{v}}^{\lrcorner}\,\mathbb{M}\,\bar{\mathbb{v}} \qquad [1.13]$$

Energies are usually quadratic in the velocities so the descriptions of energy are usually convertible into equations resembling equation [1.13]. Writing conservation of energy in this form is conducive to looking at linear combinations of the velocities. We shall see that certain linear combinations of the velocities are fundamental to collision theory.

The conservation of momentum can also be represented in this notation, but it is more difficult to construct. Tracking down the conservation of momentum will lead us to a "natural" choice of variables for collisions.

We begin our search for the conservation of momentum by refining the conservation of energy's requirements on $\mathbb{C}$.

$$\frac{1}{2}\,\mathbb{v}^{\lrcorner}\,\mathbb{M}\,\mathbb{v} = \frac{1}{2}\,\mathbb{v}^{\lrcorner}\,\mathbb{C}^{\lrcorner}\,\mathbb{M}\,\mathbb{C}\,\mathbb{v} \qquad [1.14]$$

Since $\mathbb{M}$ has zero for all of its off-diagonal elements by definition, there are no cross terms on the left side of equation [1.14]. Thus, the sum of the off-diagonal elements of $\mathbb{C}^{\lrcorner}\,\mathbb{M}\,\mathbb{C}$ must be zero so that there will also be no cross terms on the right-hand side. Hence the off-diagonal elements of $\mathbb{C}^{\lrcorner}\,\mathbb{M}\,\mathbb{C}$ are the opposite of each other. If we take the transpose of $\mathbb{C}^{\lrcorner}\,\mathbb{M}\,\mathbb{C}$, then we get

$$\left(\mathbb{C}^{\lrcorner}\,\mathbb{M}\,\mathbb{C}\right)^{\lrcorner} = \mathbb{C}^{\lrcorner}\,\mathbb{M}^{\lrcorner}\,\mathbb{C}^{\lrcorner\lrcorner} = \mathbb{C}^{\lrcorner}\,\mathbb{M}\,\mathbb{C} \qquad [1.15]$$

The off-diagonal elements of $\mathbb{C}^{\lrcorner}\,\mathbb{M}\,\mathbb{C}$ are both equal to and opposite of each other. Thus, they are zero. The off-diagonal elements cannot contribute, so the conservation of energy constrains $\mathbb{C}$ as given by the elegant equation [1.16].

$$\mathbb{C}^{\lrcorner}\,\mathbb{M}\,\mathbb{C} = \mathbb{M} \qquad [1.16]$$

The form of equation [1.16] motivates us to find the "nice" coordinate system in which $\mathbb{C}$ is diagonalized so that equation [1.16] becomes two trivial scalar equations. The eigenequation of $\mathbb{C}$ is equation [1.17].

$$\left|\lambda\mathbb{1} - \mathbb{C}\right| = \lambda^2 - 1 = 0 \qquad [1.17]$$

The eigenvalues and their corresponding eigenvectors are thus

$$\lambda_+ = +1,\ \mathbb{e}_+ = \begin{pmatrix} 1 \\ 1 \end{pmatrix};\ \lambda_- = -1,\ \mathbb{e}_- = \frac{1}{M}\begin{pmatrix} -m_2 \\ m_1 \end{pmatrix} \qquad [1.18]$$

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[5]The superscript $\lrcorner$ denotes the transpose of a matrix.

We transform from the coordinate basis where $\mathbb{C}$ is diagonalized if we use the matrix $\mathbb{T}$ constructed by using the eigenvectors as columns.

$$\mathbb{v} = \mathbb{T}\,\mathbb{V},\ \mathbb{T} \equiv \begin{pmatrix} 1 & \frac{-m_2}{M} \\ 1 & \frac{m_1}{M} \end{pmatrix} \qquad [1.19]$$

The inverse of $\mathbb{T}$ transforms from the old velocity vector to the new velocity vector.

$$\mathbb{V} = \mathbb{T}^{-1}\,\mathbb{v},\ \mathbb{T}^{-1} = \begin{pmatrix} \frac{m_1}{M} & \frac{m_2}{M} \\ -1 & 1 \end{pmatrix} \qquad [1.20]$$

Equation [1.20] is the path to our "nice" coordinate system. I will call this new basis the **centralized basis**. For contrast, I shall refer to the old basis as the **velocity basis.**

Using [1.20] we find that the new representation of velocity is

$$\mathbb{V} = \begin{pmatrix} \frac{m_1 v_1 + m_2 v_2}{M} \\ v_2 - v_1 \end{pmatrix} \qquad [1.21]$$

The collision matrix is also different in the centralized basis. We apply our transformation matrices to $\mathbb{C}$ to find its new form. In the centralized basis, it is diagonal with its eigenvalues as the diagonal elements.[6]

$$\bar{\mathbb{V}} = \mathbb{T}^{-1}\bar{\mathbb{v}} = \mathbb{T}^{-1}\,\mathbb{C}\,\mathbb{T}\,\mathbb{T}^{-1}\,\mathbb{v} = \mathbb{@}\,\mathbb{V} \qquad [1.22]$$

$$\mathbb{T}^{-1}\,\mathbb{C}\,\mathbb{T} = \mathbb{@} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad [1.23]$$

Amazingly, equation [1.22] contains both the conservation of momentum (modulo the total mass) and the equality of the approach and recession speeds. We found the conservation of momentum in the centralized basis, and we got the elasticity of the collision for free.

Though we found the conservation of momentum in the centralized basis, we never found momentum conservation in the velocity basis. The eigenvectors in one basis are the eigenvectors in any basis, so when we return to the velocity basis (or any other basis) we ought to be able to construct conservation of momentum and relative speed from the eigenvectors. $\mathbb{C}$ times its eigenvector is equal to that eigenvector's eigenvalue times the eigenvector. Using this with equation [1.16], we find

[6] @ will be used to represent the diagonalized collision matrix.

$$\mathbb{v}^{\lrcorner}\,\mathbb{M}\,\mathbb{e}_{\pm} = \mathbb{v}^{\lrcorner}\,\mathbb{C}^{\lrcorner}\,\mathbb{M}\,\mathbb{C}\,\mathbb{e}_{\pm} = \lambda_{\pm}\,\mathbb{v}^{\lrcorner}\,\mathbb{C}^{\lrcorner}\,\mathbb{M}\,\mathbb{e}_{\pm} \qquad [1.24]$$

When equation [1.24] is used with the eigenvalues and eigenvectors for the point particle given in equations [1.18], it yields

$$m_1 v_1 + m_2 v_2 = \mathbb{v}^{\lrcorner}\,\mathbb{M}\,\mathbb{e}_{+} = \lambda_{+}\,\mathbb{v}^{\lrcorner}\,\mathbb{C}^{\lrcorner}\,\mathbb{M}\,\mathbb{e}_{+} = m_1 \bar{v}_1 + m_2 \bar{v}_2 \qquad [1.25]$$

$$v \equiv v_2 - v_1 = \mathbb{v}^{\lrcorner}\,\mathbb{M}\,\mathbb{e}_{-} = \lambda_{-}\,\mathbb{v}^{\lrcorner}\,\mathbb{C}^{\lrcorner}\,\mathbb{M}\,\mathbb{e}_{-} = -\left(\bar{v}_2 - \bar{v}_1\right) = -\bar{v} \qquad [1.26]$$

Equation [1.25] shows that momentum is conserved in the point particle collision, and equation [1.26] shows elasticity in the point particle collision. It is the eigenvalues and eigenvectors of the collision matrix that give information about the relative velocity and momentum.

We found the centralized basis by diagonalizing the collision matrix of the point particle problem, but the centralized basis is a useful construction in many other collision problems. If we define the centralized basis as the basis in which the components of $\mathcal{V}$ play the same role that they did previously, the centralized basis becomes generally useful. To promote the components' importance, I will properly define them.

$$\mathcal{V} = \begin{pmatrix} \dfrac{m_1 v_1 + m_2 v_2}{M} \\ v_2 - v_1 \end{pmatrix} \equiv \begin{pmatrix} V \\ v \end{pmatrix} \qquad [1.27]$$

The **relative velocity**, v, is the difference between the velocities. We have already discussed the relative velocity because of its close link to elasticity. It is defined in equation [1.28].

$$v \equiv v_2 - v_1 \qquad [1.28]$$

The relative velocity is the second component of the centralized velocity vector.

Given the particles' positions x_1 and x_2, the **position of the center of mass**, X, is defined as

$$X \equiv \frac{m_1 x_1 + m_2 x_2}{M} \qquad [1.29]$$

If we implicitly differentiate equation [1.29] with respect to time, we find the **center of mass velocity**.

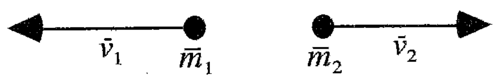
$$V \equiv \frac{m_1 v_1 + m_2 v_2}{M} \qquad [1.30]$$

The center of mass velocity is the first component of the centralized velocity vector.

With these definitions, the transformation matrices between the centralized basis and the velocity basis are the same even if the collision is not elastic. It is not yet guaranteed, however, that the collision matrix will always be diagonalized when the centralized basis is defined in this manner. We only know that the collision matrix for the elastic point particle collision is diagonalized. It will remain diagonalized in the second example, however.

The second example requires generalization of the conservation of energy and the conservation of momentum to allow the particles to change mass during the collision. Kinetic energy and momentum are still conserved, however.

[1.3] -- The point particles approach each other before the collision.



[1.4] -- The point particles depart after the collision. Note they have changed mass during the collision.

Because we are concerned with non-relativistic collisions, the total mass is also conserved. Since the total mass is fixed, we can consider any change in the masses as an exchange of mass from one particle to the other. Let Δm represent the loss in mass from particle one and the gain in mass of particle two. The final masses of the particles are

$$\bar{m}_1 = m_1 - \Delta m, \; \bar{m}_2 = m_2 + \Delta m \qquad [1.31]$$

Of course, Δm could be negative so that particle one would gain mass and particle two would lose mass.

In the centralized basis, the conservation of energy has essentially the same form as in the velocity basis. The conservation of energy is represented by equation [1.32] when the mass is allowed to change and the velocities are transformed by equation [1.19]. Remember that the transformation matrices are dependent on the masses, so they also change after the collision.

$$\frac{1}{2}\,\mathbb{V}^{\dashv}\mathbb{T}^{\dashv}\,\mathbb{M}\;\mathbb{T}\;\mathbb{V} = \frac{1}{2}\,\overline{\mathbb{V}}^{\dashv}\overline{\mathbb{T}}^{\dashv}\,\overline{\mathbb{M}}\;\overline{\mathbb{T}}\;\overline{\mathbb{V}} \qquad [1.32]$$

Or

$$\frac{1}{2}\,\mathbb{V}^{\dashv}\,\mathcal{M}\,\mathbb{V} = \frac{1}{2}\,\overline{\mathbb{V}}^{\dashv}\,\overline{\mathcal{M}}\,\overline{\mathbb{V}} \qquad [1.33]$$

Where $\mathcal{M}$ is the **centralized mass matrix**:

$$\mathcal{M} \equiv \mathbb{T}^{\dashv}\,\mathbb{M}\;\mathbb{T}\,, \; \mathcal{M} \equiv \begin{pmatrix} M & 0 \\ 0 & \mu \end{pmatrix} \qquad [1.34]$$

And μ is the **reduced mass**,

$$\mu \equiv \frac{m_1 m_2}{m_1 + m_2} \qquad [1.35]$$

Equation [1.33] is the conservation of energy in the centralized basis.

The conservation of momentum becomes (with reference to equation [1.27]),

$$MV = \bar{M}\bar{V} \qquad [1.36]$$

The total mass doesn't change, so we are left with

$$V = \bar{V} \qquad [1.37]$$

Equation [1.37] forces us to choose the first row of our collision matrix as given in equation [1.38].[7]

$$\mathbb{C} = \begin{pmatrix} 1 & 0 \\ \lfloor \mathbb{C} \rfloor_{2,1} & \lfloor \mathbb{C} \rfloor_{2,2} \end{pmatrix} \qquad [1.38]$$

The bottom row of the centralized collision matrix is also constrained, not by conservation of momentum, but by the role of the observer. The collision matrix ought to apply regardless of the frame of the observer. The bottom row of the collision matrix represents equation [1.39].

$$\bar{v} = \lfloor \mathbb{C} \rfloor_{2,1} V + \lfloor \mathbb{C} \rfloor_{2,2} v \qquad [1.39]$$

Suppose an observer finds the collision matrix in a particular inertial frame. If she changes frames to a frame moving with constant velocity with respect to the first frame, she expects there to be a change in the center of mass velocity, but no change in the relative velocity.[8] If equation [1.39] is to hold in the new frame it must be the case that $\lfloor \mathbb{C} \rfloor_{2,1}$ vanishes.

We also have more information about the fourth matrix element. $\lfloor \mathbb{C} \rfloor_{2,2}$ has the same magnitude as the ratio of the outgoing relative speed to the incoming relative speed. In other words, its magnitude is the coefficient of restitution!

From the preceding three paragraphs, we know that if the total momentum and total mass are conserved and the observer's frame doesn't affect the collision (in the low velocity limit) $\mathbb{C}$ is always diagonal in the centralized basis. Moreover, we know what the eigenvalues of the collision matrix are.

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[7]Brackets open at the top indicate a matrix element, e.g., $\lfloor \mathbb{M} \rfloor_{1,2}$ is the first-row, second-column element of the matrix $\mathbb{M}$.

[8]Remember that we are not using special relativity: the velocities are small enough so that we may use Galileo's rather than Einstein's velocity addition. There is of course a minute change in the relative velocity according to special relativity, but we will neglect it, just as we have been neglecting changes in mass due to changes in binding energies, etc.

$$\mathbb{C} = \begin{pmatrix} 1 & 0 \\ 0 & -e \end{pmatrix} \quad [1.40]$$

The coefficient of restitution is e. This result gives us all the constraints specified without the conservation of energy.

The conservation of energy determines the value of the coefficient of restitution in equation [1.40]. If the collision matrix from equation [1.40] is used in the conservation of energy, equation [1.33], the result is

$$\frac{1}{2}\mathbb{V}^{\lrcorner}\mathbb{M}\mathbb{V} = \frac{1}{2}\mathbb{V}^{\lrcorner}\mathbb{C}^{\lrcorner}\overline{\mathbb{M}}\mathbb{C}\mathbb{V} \quad [1.41]$$

Equations [1.31], [1.34] and [1.35] determine $\mathbb{M}$ and $\overline{\mathbb{M}}$.

$$\mathbb{M} = \begin{pmatrix} M & 0 \\ 0 & \mu \end{pmatrix} \quad [1.42]$$

$$\overline{\mathbb{M}} = \begin{pmatrix} \overline{M} & 0 \\ 0 & \overline{\mu} \end{pmatrix} = \begin{pmatrix} M & 0 \\ 0 & \dfrac{(m_1 - \Delta m)(m_2 + \Delta m)}{M} \end{pmatrix} \quad [1.43]$$

We expand equation [1.41].

$$\frac{1}{2}MV^2 + \frac{1}{2}\mu v^2 = \frac{1}{2}MV^2 + \frac{1}{2}\overline{\mu}e^2 v^2 \quad [1.44]$$

Or

$$e^2 = \frac{\mu}{\overline{\mu}} \quad [1.45]$$

We must pause because it is possible for equation [1.45] to be undefined. If the amount of mass exchanged is just right, then $\overline{\mu}$ may be zero. Looking back to [1.44], we see that if all the mass were exchanged, the relative velocity would be zero, so we choose the coefficient of restitution to be zero when all the mass is exchanged.[9] Otherwise, we want the particles to end the collision moving apart, so we take the positive root of equation [1.45].

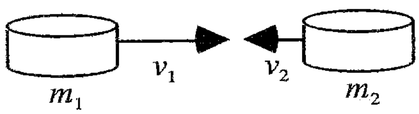
$$e = \sqrt{\frac{\mu}{\overline{\mu}}} = \sqrt{\frac{m_1 m_2}{\overline{m}_1 \overline{m}_2}} = \sqrt{\frac{m_1 m_2}{(m_1 - \Delta m)(m_2 + \Delta m)}} \quad [1.46]$$

$$e = 0 \text{ } if \text{ } m_1 = \Delta m \text{ } or \text{ } m_2 = -\Delta m$$

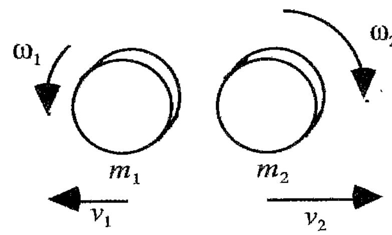
[9]When the coefficient of restitution is zero, the collision is said to be totally inelastic.

Equations [1.40] and [1.46] constitute the solution to the second example. In this example, we also learned that if the conservation of kinetic energy, conservation of momentum, and conservation of mass apply in a low velocity setting, then it must be the case that the centralized collision matrix is diagonalized with 1 and the negative of the coefficient of restitution as its eigenvalues.

The final collision to be considered in this chapter involves energy being transferred from translational energy to rotational energy during the collision. In this situation, the coefficient of restitution is not constant; it depends on the velocities.



[1.5] -- The two hockey pucks approach each other by sliding.



[1.6] -- The hockey pucks separate by rolling without slipping.

Consider two cylindrical hockey pucks that slide towards each other on a frictionless surface (figure [1.5]). The pucks collide and then separate. When they separate, they roll without slipping (figure [1.6]).

The conservation of energy must include the rotational kinetic energy of the pucks after the collision.

$$\frac{1}{2}m_1v_1^{\,2}+\frac{1}{2}m_2v_2^{\,2}=\frac{1}{2}m_1\bar{v}_1^{\,2}+\frac{1}{2}I_1\bar{\omega}_1^{\,2}+\frac{1}{2}m_2\bar{v}_2^{\,2}+\frac{1}{2}I_2\bar{\omega}_1^{\,2} \qquad [1.47]$$

Where I is the moment of inertia of the puck and ω is the angular velocity.[10]

The pucks roll without slipping, so the angular velocity is the linear velocity divided by the radius.

It is well known that the moment of inertia of a cylinder rotating about its axis is $m\,r^2 \div 2$.[11] By substituting for the angular momentum and the moment of inertia in equation [1.47], equation [1.48] is produced.

$$\frac{1}{2}m_1v_1^{\,2}+\frac{1}{2}m_2v_2^{\,2}=\frac{3}{4}m_1\bar{v}_1^{\,2}+\frac{3}{4}m_2\bar{v}_2^{\,2} \qquad [1.48]$$

Equation [1.48] can easily be represented in matrix notation because it is identical to the ordinary conservation of energy equation, except that the right side is multiplied by three fourths rather than a half.

[10]We suppose that the pucks do not gain any gravitational potential by hopping up on their sides, or at least they are moving quickly enough that this potential is negligible.

[11]P. Tipler. *Physics for Scientists and Engineers,* 3rd ed. (New York: Worth Publishers, 1991) pp. 233.

$$\frac{1}{2}\mathbb{V}^{\dashv}\mathbb{M}\,\mathbb{V}=\frac{3}{4}\overline{\mathbb{V}}^{\dashv}\mathbb{M}\,\overline{\mathbb{V}}=\frac{3}{4}\mathbb{V}^{\dashv}\mathbb{C}^{\dashv}\mathbb{M}\,\mathbb{C}\,\mathbb{V} \qquad [1.49]$$

As I noted above, energy considerations are absent from equation [1.40] so it applies to this third problem as well. Using equation [1.40] and the conservation of energy (equation [1.50]) we find

$$\frac{1}{2}\mathbb{V}^{\dashv}\mathbb{M}\,\mathbb{V}=\frac{3}{4}\mathbb{V}^{\dashv}\mathbb{C}^{\dashv}\mathbb{M}\,\mathbb{C}\,\mathbb{V} \qquad [1.50]$$

Becomes

$$\frac{2}{3}\left(MV^2+\mu v^2\right)=MV^2+\mu e^2 v^2 \qquad [1.51]$$

Equation [1.51] is solved for the coefficient of restitution to give

$$e=\frac{2}{3}-\frac{1}{3}\frac{M}{\mu}\left(\frac{V}{v}\right)^2 \qquad [1.52]$$

The collision matrix is thus

$$\mathbb{C}=\begin{pmatrix} 1 & 0 \\ 0 & -\frac{2}{3}+\frac{1}{3}\frac{M}{\mu}\left(\frac{V}{v}\right)^2 \end{pmatrix} \qquad [1.53]$$

The colliding hockey puck problem is solved.

Previously, I argued that the collision matrix should be the same regardless of observer's frame. Obviously, equation [1.53] does not follow this rule. What happened? Well, along the way, I made a tacit assumption that the observer was in the same frame as the floor where the pucks are rolling. Since the floor is part of the dynamics of the collision, the frame of the floor does matter. The collision matrix is still the same in all frames if the V in equation [1.53] is interpreted as the difference in velocities of the floor and the center of mass of the pucks.

Even with this distinction, the coefficient is a function of the velocities. If we couldn't observe the hockey pucks' rotation, then we would have quite a problem. Not only would we be unable to account for the loss of energy, but the amount of energy lost would change with the velocities. Rotation is a very tricky hidden variable.

In this chapter we have solved three problems: the point particle collision, the exchanging mass point particles, and the hockey puck collision. In each of these problems, we found new insights and techniques useful for collision mechanics.

In the first example, the point particles kept the same mass and conserved kinetic energy. The particles' center of mass velocity was conserved, while their relative velocity kept the same magnitude, but changed sign. This elastic collision led us to wonder if there might be a notation that emphasizes these conserved quantities. Because the conserved velocities were linear combinations of the particles' velocities, we chose a new notation using matrices and vectors. The collision matrix, which "collides the velocities," has the center of mass velocity and the relative velocity as eigenvectors.

In the second example, I defined the centralized basis as the basis where the components of velocity were the center of mass velocity and the relative velocity. If 1) momentum was conserved, 2) the total mass was conserved, and 3) the collision was not dependent on the observer's frame, the collision matrix in the centralized basis is always given by

$$\mathbb{C} = \begin{pmatrix} 1 & 0 \\ 0 & -e \end{pmatrix} \qquad [1.54]$$

This is the most efficient statement of all constraints on the collision except for the conservation of energy.

The only missing information in equation [1.54] is the value of the coefficient of restitution. Determining this value is the job of the conservation of energy. This is why the loss of energy to hidden variables causes difficulty. If energy conservation cannot be used in some form, then the coefficient of restitution cannot be determined and the final velocities cannot be found.

The first two examples explored the conservation of kinetic energy. The conservation in the point particle collision gave elasticity, but the second example revealed that if the particle masses changed, the collision can be inelastic even when kinetic energy is conserved.

In the third example, the conservation of kinetic energy was modified to include rotational kinetic energy. The collision was inelastic, even though the hockey pucks' masses were constant; this was possible because even though kinetic energy was conserved, the kinetic energy moving the centers of mass of the pucks was not. In fact, the coefficient of restitution depended on the velocities of the pucks relative to the floor. If rotational kinetic energy were a hidden variable, the coefficient of restitution would be exceptionally difficult to find experimentally, because it would vary not only with the masses but also with the velocities.

There are many tricks for calculating the results of collisions. The results are determined by two factors.

The first factor is that with three assumptions (conservation of momentum, dynamical insignificance of the observer's frame of reference, and conservation of total mass) it was possible to prove that the center of mass velocity of the particles must be conserved. It was also found to be linearly independent of the relative velocity.

The second constraint is the conservation of energy, which gives the value of the coefficient of restitution. The coefficient of restitution governs the relative velocity. A collision problem is solved in two steps: first, the conservation of momentum gives the center of mass velocity, and then conservation of energy gives the relative velocity.

Chapter Two: Particles on Strings

In this second chapter, I move beyond the traditional collisions to discuss the simplest of the models with internal degrees of freedom, **the particles-on-strings model.** It correctly predicts the coefficient of restitution for the cylinder collision, but is not sufficiently deep to capture the sphere collision. The model is interesting in its own right, so some of the discussion covers general features of the model not directly relevant to the cylinders and spheres.

The colliding objects, or **clumps**, are constructed from point particles, as dealt with in chapter one. These point particles are "tied together with strings" to form the clumps: each clump is a one-dimensional array of point particles constrained so that the distance between each pair of particles is less than a certain constant, the **string length**. Otherwise, the point particles move as free particles and may collide with each other.

[2.1] -- The clumps are constructed from point particles tied together with strings.

A particle in this model has two forms of interaction with other particles available. The first is the ordinary collision, or **bang**, as dealt with in chapter one. The collision of point particles is considered to be an instantaneous interaction occurring during the instant in which two particles inhabit the same position. As in chapter one, the collision is governed by

$$\bar{\mathbb{v}} = \mathbb{C}\,\mathbb{v} \qquad [2.1]$$

With the collision matrix

$$\mathbb{C} = \frac{1}{M}\begin{pmatrix} m_1 - m_2 & 2\,m_2 \\ 2\,m_1 & m_2 - m_1 \end{pmatrix} \qquad [2.2]$$

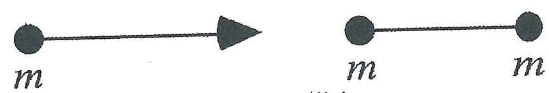
The banging of the particles ensures that they remain in the same order as they began. Instead of passing each other, they bang.

The particles must also be kept from exceeding the string length. The interaction that keeps them together I will call a **twang**. If two particles move apart so that their distance reaches the string length, they twang. The twang does not violate conservation of kinetic energy or conservation of momentum, so for two adjacent particles

$$\frac{1}{2}m_1 v_1^{\,2} + \frac{1}{2}m_2 v_2^{\,2} = \frac{1}{2}m_1 \bar{v}_1^{\,2} + \frac{1}{2}m_2 \bar{v}_2^{\,2} \qquad [2.3]$$

$$m_1 v_1 + m_2 v_2 = m_1 \bar{v}_1 + m_2 \bar{v}_2 \qquad [2.4]$$

These equations are identical to the equations governing a bang, so the particles have the same outgoing velocities that they would have had they banged. Thus, we calculate the velocities after a twang the same way as after a bang: we apply the collision matrix.



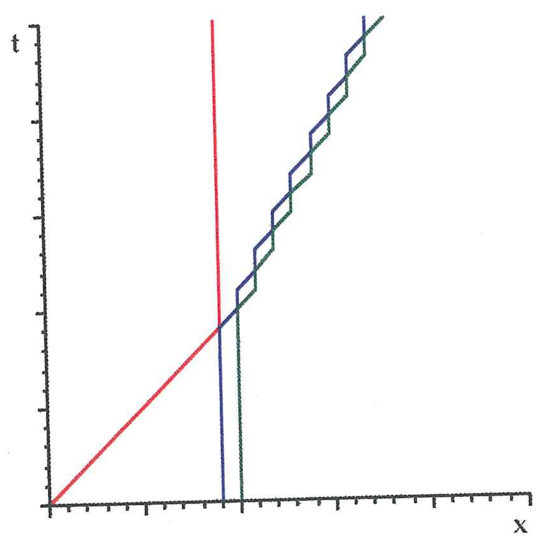
[2.2] -- The simplest clump collision.

The clumps interact through the interaction of their particles. When the clumps collide what actually happens is that their constituent particles collide.

The simplest possible problem with more than just two point particles is the collision of a moving point particle with a motionless two-particle clump. The particles in the clump are separated by less than the string length, but they are separated by a non-zero distance. I will use n to denote the number of particles in the smaller of the two clumps and N to denote the number of particles in the larger clump, so in this case, n=1 and N=2. To simplify further, all the point particles have the same mass.

The clumps approach each other and the leading particles bang. Afterwards, the leading particles are moving apart from each other, so the next collision is a bang between the two particles in the clump. Afterwards, the two particles in the clump twang. Figure [2.3] is a space-time diagram of the collision.

In figure [2.3] the total energy and total momentum are conserved because at any time, one particle is moving with the same velocity as the incoming particle while the other two particles are standing still. If m is the mass of the particles and v is the velocity of the moving particle, the energy (E) and the momentum (P) are constant.

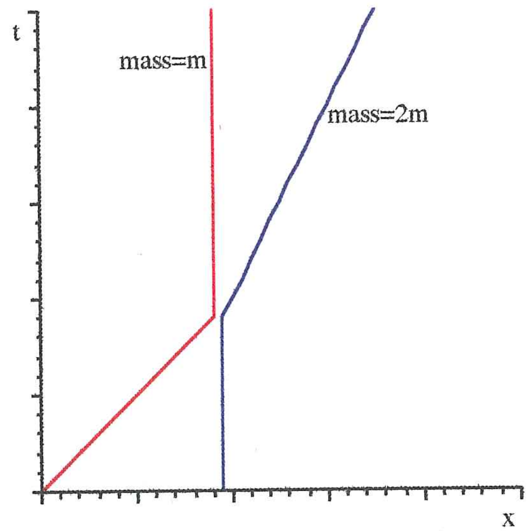


[2.3] -- Space-time diagram of the particles in the n=1, N=2 collision with no internal energy. Equal particle masses. Equal string lengths. e=0.5.

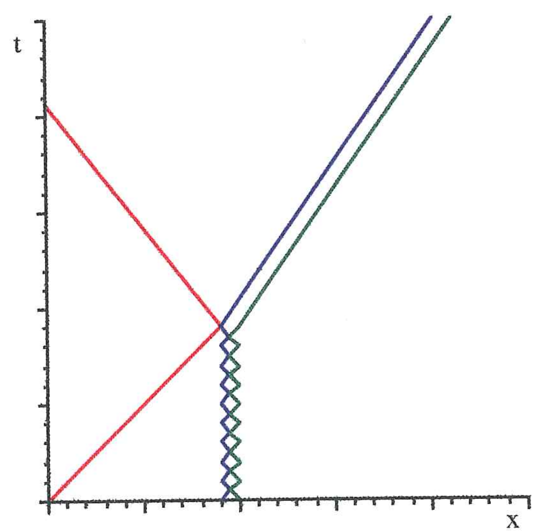
$$E = \frac{1}{2} m v^2 \qquad [2.5]$$

$$P = mv \qquad [2.6]$$

If, on the other hand, the internal motion of the right clump is hidden, and only the motion of the centers of mass of the clumps can be measured, then the space-time diagram would be like the one given in figure [2.4].



[2.4] -- Space-time diagram of the centers of mass in the n=1, N=2 collision with no internal energy. e=0.5.



[2.5] -- Space-time diagram of the particles in the n=1, N=2 collision with the internal energy of equal speeds. Equal particle masses. Equal string lengths. e=2.

In figure [2.4] the momentum is still given by equation [2.6], and the total energy is still given by equation [2.5]. The observable energy is also given by equation [2.5] before the collision, but after the collision it is given by equation [2.7].

$$E_{CM} = \frac{1}{2}\left(2m\right)\left(\frac{1}{2}v\right)^2 = \frac{1}{4}mv^2 \qquad [2.7]$$

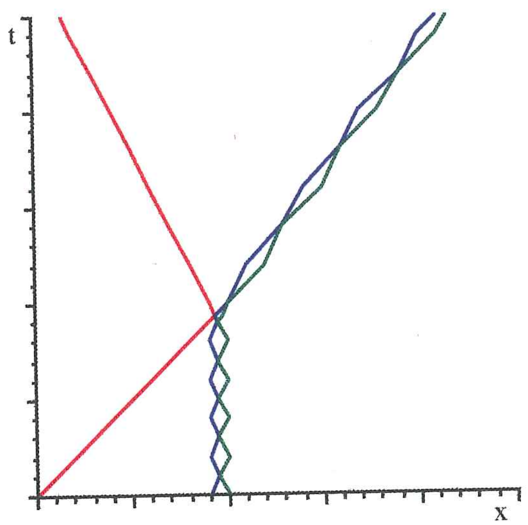
Half of the energy becomes hidden during the collision! The coefficient of restitution of this inelastic collision is 0.5 even though all the particles collide elastically.

In the first collision, all the energy was observable before the collision, but what if the particles in the right clump were moving? The particles in the right clump may lose some (or as in figure [2.5], all) of their internal energy to observable energy. Of course, this increases the total observable energy, so the coefficient of restitution is greater than one. It is 2 for the collision in figure [2.5].

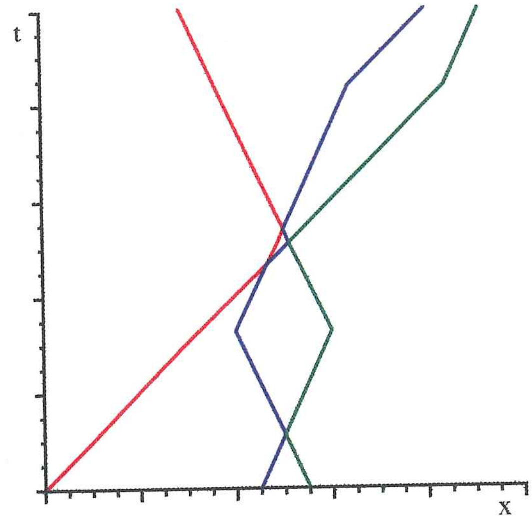
Decreasing the amount of internal energy decreases the coefficient of restitution. In figure [2.6] the internal velocities are reduced by a half of those in figure [2.5], so the internal energy is decreased to a fourth of its previous value.

It is not always the case that the leading particles collide only once with the leading particle of the clump. In Figure [2.7] they collide twice.

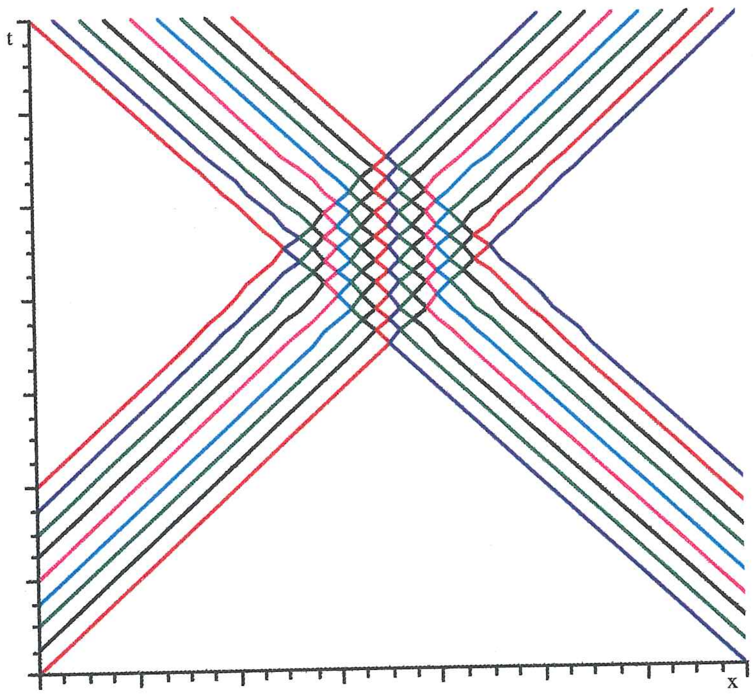
There are many complications introduced by allowing the clumps to begin the collisions with internal energy. However, we expect our real cylinders and spheres to be static before the collision. Whatever internal energy they might have would be distributed as heat, not as coherent wave-like phenomena.



[2.6] -- Space-time diagram of the particles in the n=1, N=2 collision with less internal energy. Equal particle masses. Equal string lengths. e=1.25.



[2.7] -- Close-up of space-time diagram of the particles in the n=1, N=2 collision with some internal energy. The leading particles collide twice rather than once with this velocity combination. e=1.19.

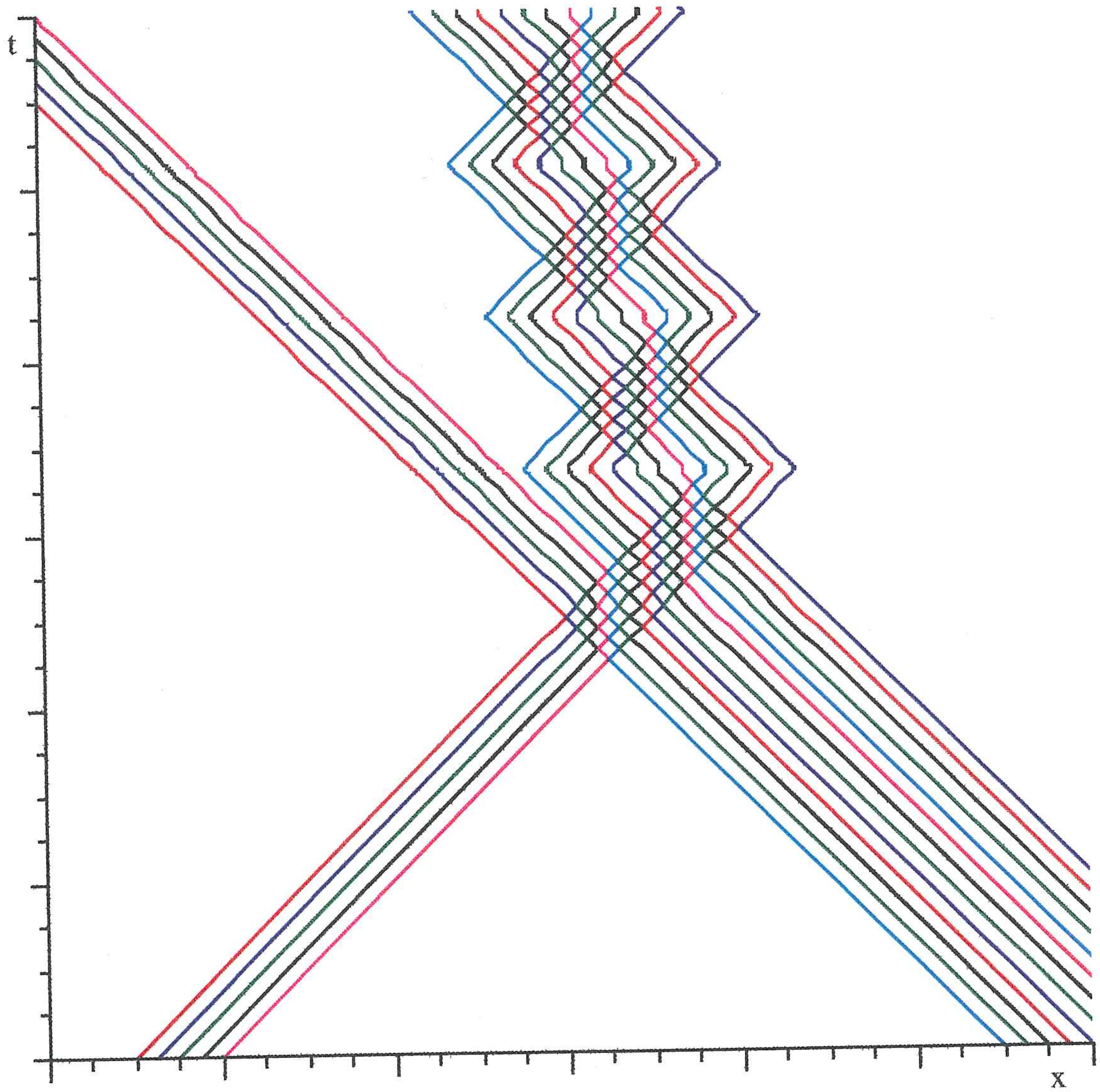


[2.8] -- The space-time diagram for a collision between two nine-particle clumps. Equal particle masses. Equal string lengths. e=1.

Colliding objects that are already ringing are interesting, but we will assume from now on that all of our clumps enter the collision with no internal energy.

We can, of course collide larger clumps. The True BASIC program in appendix A collides these larger clumps. Figure [2.8] is a collision between two clumps of equal length (n=N). The collision is elastic.

The collision of unequal length clumps is inelastic. Figure [2.9] is a collision between an n=5 clump and an N=13 clump. There are five collisions between the leading particles before the clumps separate. It is during these five collisions that all the momentum and energy is exchanged. Because the particle masses are equal, the particles can be traveling left or right, but there is only one speed they can have. During the five collisions, five of the particles in the five-particle clump are turned around, and five of the particles in the thirteen-particle clump are turned around.



[2.9] -- Space-time diagram of the particles in the n=5, N=13 collision. Equal particle masses. Equal string lengths. e=5/13.

Because the other eight particles in the thirteen-particle clump are still traveling in their original direction, there is internal energy left in the thirteen-particle clump. The thirteen-particle clump has five changed-velocity particles and eight unchanged-velocity particles after collision, so its average (center of mass) velocity is (8-5)/13 = 3/13 times its original velocity.

The coefficient of restitution is thus

$$e = \left| \frac{5v + (-8v)}{13} - (-v) \right| \div 2v = \frac{5}{13} \qquad [2.8]$$

When other lengths of clumps collide, the same exchange of energy quanta occurs. There are n collisions between the leading particles before the smaller clump escapes. The smaller clump has no internal energy afterwards because each particle has reversed direction, and the velocity of the smaller clump is the opposite of what it was before collision. The larger clump, however, still has N-n particles moving in their original direction. The absolute value of the difference in center-of-mass velocities of the clumps is the coefficient of restitution.

$$e = \left| \frac{nv - (N-n)v}{N} - (-v) \right| \div 2v = \frac{2n}{2N} = \frac{n}{N} \qquad [2.9]$$

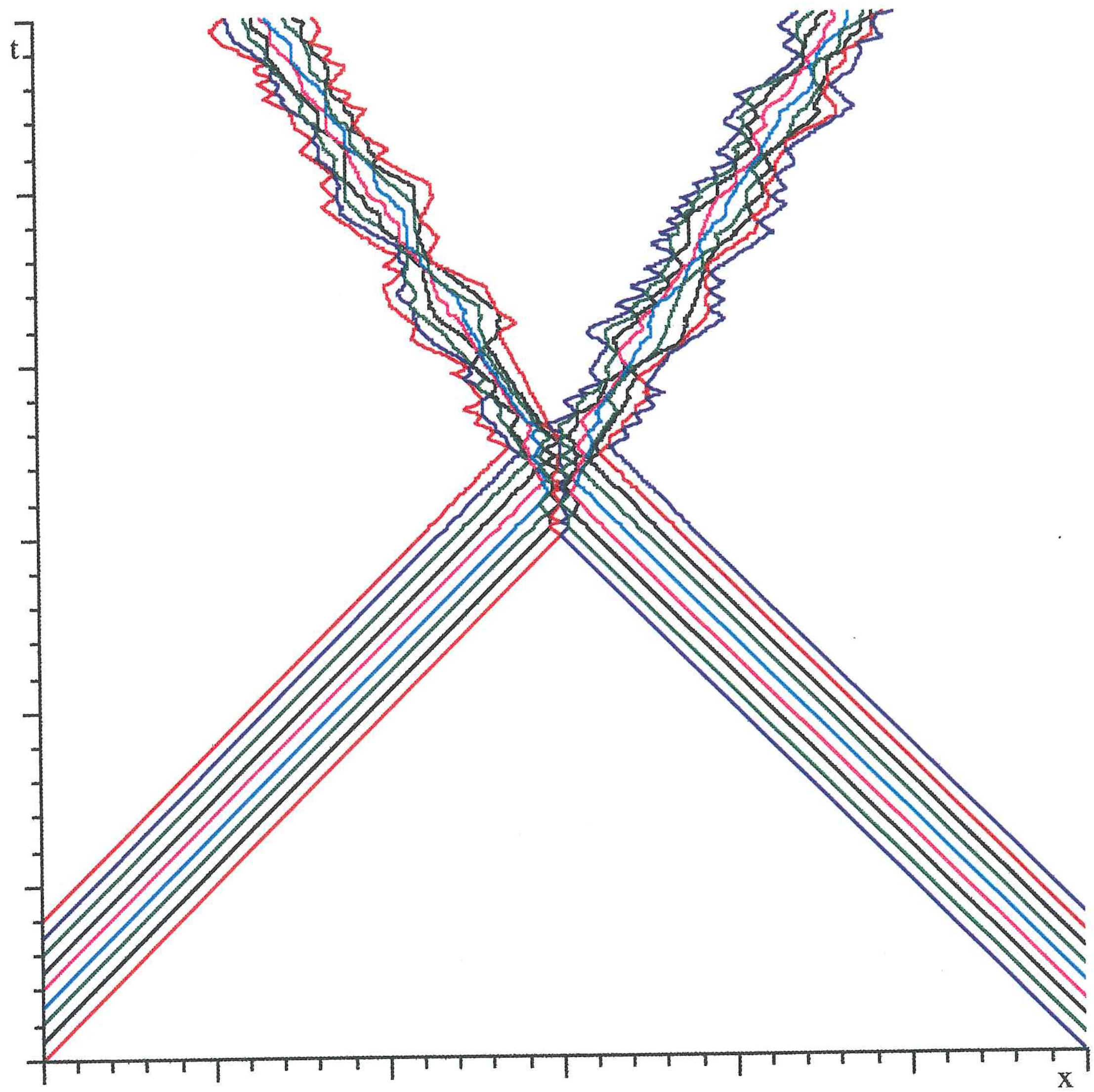
Equation [2.9] gives the coefficient of restitution for any collision between clumps where all the string lengths and particle masses are equal. What kind of real world object does this kind of clump model? Well, it has uniform density and compressibility along the axis of collision, so the real world object must be uniform along this dimension as well. The equal string length, equal mass clumps resemble cylinders (or other tube like solids). Thus we have found our cylinders using the string model. Equation [2.9] gives the correct value of the coefficient of restitution for colliding cylinders when the particles are equally spaced so the number of particles is proportional to the length.

Many simulations with different values of N and n were run using the computer program. The results of these runs are given in table [2.10]. Notice that they agree exactly with the values predicted by equation [2.9].

# of particles	5	7	9	11	13	15
5	1.0000	0.71429	0.55556	0.45455	0.38462	0.33333
7		1.0000	0.77778	0.63636	0.53846	0.46667
9			1.0000	0.81818	0.69231	0.60000
11				1.0000	0.84615	0.73333
13					1.0000	0.86667
15						1.0000

[2.10] -- The coefficients of restitution found with particles-on-string modeling of cylinder collisions. The collisions were simulated with the code in Appendix A.

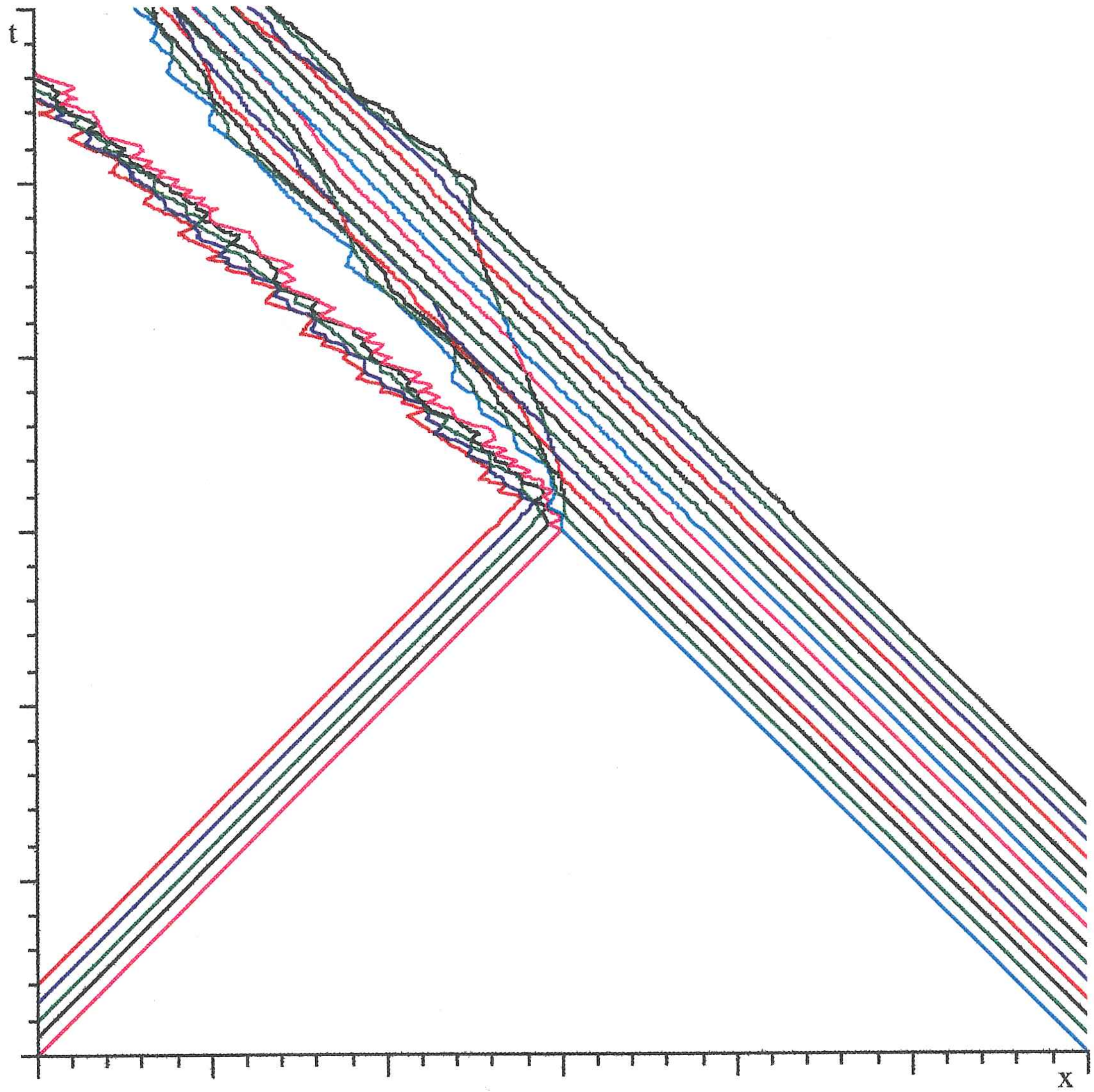
The sphere collision is not as easily modeled as the cylinder collision. There are a number of choices to make about the structure of a sphere-resembling clump. The first method is to take clumps identical to those used for the cylinders, but change their masses so that their mass is spherically distributed. The second option is to make the string lengths different, but make all the masses within a particular clump equal, so that again the mass is spherically distributed. The third option is to choose one particular string length and mass configuration and the scale it up and down without changing the number of particles. I have found that none of these modeling systems accurately model the sphere.



[2.11] -- A collision of two equal sphere-like clumps that differ from the cylinder clumps by mass distribution. e=0.64.

The first method of simulating spheres consists of making the spheres have the same string lengths and number of particles as the corresponding cylinder, but a different mass distribution. A collision of equal radii "spheres" constructed using this model is shown in figure [2.11].

We expect that the spheres will collide elastically regardless of the relative sizes of the spheres, yet this clump model doesn't even predict elastic collisions for equal radii "spheres". It predicts even less elasticity for the collision of unequal radii "spheres".

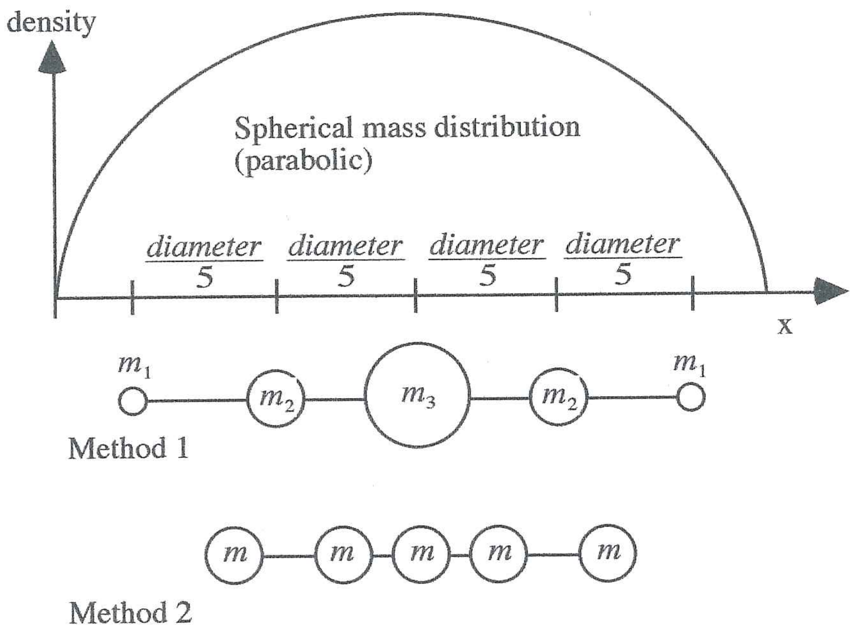


[2.12] -- A collision of two unequal sphere-like clumps that differ from the cylinder clumps by mass distribution. e=0.14.

Real spheres collide more elastically than cylinders, but this model's "spheres" collide less elastically than cylinders.

We can see easily why the "sphere" clumps are less elastic. The cylinder clumps were easily analyzed because the particles only had two velocities available, left and right with the same speed either way. In the collision of these "spheres", there are many different velocity possibilities, so it would be an absolute miracle if it worked out so neatly that there was no internal energy after the collision.

To remove some of the freedom in the particle velocities, I tried a second sphere model. In this model, all the particles within a particular clump have the same mass, but the string lengths are varied so that the mass profile of a sphere is simulated. In the equal radii "spheres" collision, we have elasticity as we would like. This is because the same collisions occur here as in the equal cylinders' collision, only the timing is different.



[1.7] -- In the first method of simulating spheres, the string lengths are held constant and equal while the masses are varied. In the second method, the masses are held equal and constant while the string lengths are varied.

What happens, though, when we collide unequal radii "spheres" of this kind? A sphere's mass scales as the radius cubed, so when we reduce the radius by a half, we have to reduce the mass by an eighth. The number of particles is reduced by an eighth while the radius is reduced by a half. This is identical, however, to the cylinder collision where one of the cylinders is eight times as long as the other, except the string lengths are different. What ends up happening is that the string lengths do not change the elasticity at all, they only change the timing of the collisions.

The final modeling technique, scaling one particular "sphere" model (that was constructed as in the first method) didn't work either. Ending up with zero internal energy is quite difficult. With this third model, the spheres were less elastic than the cylinders again.

Tables [2.13], [2.14], and [2.15] give the values for the coefficients of restitution found with the first and third sphere models. As explained above, the second model reduces to a cylinder collision, and thus obeys the cylinder collision formula. All three models do not behave as spheres do: the particles-on-strings model doesn't capture the essence of the sphere collision.

# of particles	5	7	9	11	13	15
5	1.0000	0.60000	0.40000	0.31133	0.26667	0.30000
7		1.0000	0.40000	0.40000	0.40000	0.30939
9			1.0000	0.40000	0.40000	0.40000
11				1.0000	0.40000	0.40000
13					1.0000	0.40000
15						1.0000

[2.13] -- The coefficients of restitution for the first type of sphere model. This model is identical to the cylinders in terms of number of particles and string lengths, but the masses of the particles are spherically distributed.

total mass	5	7	9	11	13	15
5	1.000000	0.445428	0.324295	0.233058	0.613262	0.733975
7		1.000000	0.600000	0.672162	0.305083	0.209138
9			1.000000	0.600000	0.400684	0.315789
11				1.000000	0.600000	0.439236
13					1.000000	0.600000
15						1.000000

[2.14] -- Some coefficients of restitution for the third type of sphere model. This model uses one standard "sphere" model constructed from a five-particle clump scaled by varying the masses and string lengths. The mass of the "spheres" varied as the third power of the "radius".

total mass	5	7	9	11	13	15
5	0.777778	0.507338	0.494501	0.339721	0.493722	0.391891
7		0.777778	0.555555	0.635736	0.289083	0.358112
9			0.777777	0.758800	0.562234	0.540936
11				0.777778	0.555555	0.513633
13					0.777778	0.777778
15						0.777778

[2.15] -- Some coefficients of restitution for the third type of sphere model. This model uses one standard "sphere" model constructed from a nine-particle clump scaled by varying the masses and string lengths. The total mass of the "spheres" varied as the third power of the "radius".

In this chapter, particles on strings were used to model systems with internal degrees of freedom. This style of model is rarely used, probably because it requires the use of discontinuous, numerical mathematics rather than the smooth analysis of calculus-based mechanics. Even so, this model is deep enough to provide the correct prediction for the collision of two cylinders. For the cylinder collision, the coefficient of restitution is the ratio of the lengths of the cylinders, shorter to longer.

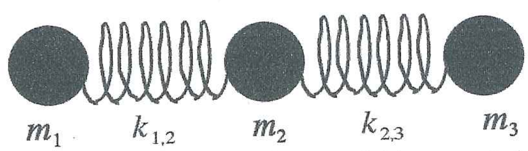
The particles-on-strings model does not seem to be capable of capturing the collision of two spheres. Many reasonable approaches to modeling spheres are possible with the particles-on-strings model, yet none predict the spheres to be more elastic, or even as elastic as the cylinders. The essential problem is that the cylinder collision is the simplest to model using particles on strings. It cleanly solved because it contains only two types of particles, those traveling to the left and those traveling to the right. As soon as more complications, e.g., variations in mass, velocities, or string lengths, are introduced, the symmetry of the problem is lost. The seemingly chaotic outcome is certainly not capable of disposing of all internal energy in an orderly fashion. Thus, the model is not capable of producing the elastic collisions we seek in a model of the spherical collision.

Chapter Three: Particles on Springs

In chapter two, analysis of models constructed from particles on strings found the expected results for the cylinder collision but not for the sphere collision. The reason that the model for the cylinders worked out so neatly was that when the point particles collided there were only two possible outgoing velocities: left or right with the same speed. The limit on the velocities reduced the collision to a counting problem. The spherical models, on the other hand, had a different mass for each particle so many different outgoing velocities were possible. It would have been miraculous if all of these different velocities had canceled out to leave no internal energy after the collision. This miracle did not occur, and the spherical models retained internal energy after the collision.

A spherical model should not gain internal energy during the collision, so it must be highly ordered to reconvert the vibrational energy produced by the collision back into translational energy. With this in mind, we might impose a constraint to reduce the number of possible velocities.

The particles on strings model was essentially a free particle model. Other than the internal collisions that occurred when the particles occupied the same space or reached the string length, there was no interaction between the particles. It is rare that real physical problems enjoy this kind of freedom, so we are led to consider whether we might introduce an interaction between the point particles that would also give order to the velocities.



[3.1] -- The particles-on-strings model is upgraded to a particles-on-springs model.

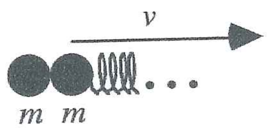
The obvious first choice for a particle interaction is a linear attraction on the separation distance between the particles: a Hooke's law spring force. Thus we are led to consider a new model, particles on springs.

As always, we presume that the point particles collide in instantaneous collisions obeying the collision matrix from chapter one.

$$\mathbb{C} = \frac{1}{M}\begin{pmatrix} m_1 - m_2 & 2\,m_2 \\ 2\,m_1 & m_2 - m_1 \end{pmatrix} \qquad [3.1]$$

We'll choose to make the springs strong enough that the particles within a clump do not hit each other: the only collisions between point particles are the collisions between the two leading particles of different clumps. The springs, of course, are massless.

[3.2] -- Before the point particle hits the clump it has some velocity.



[3.3] -- Immediately after the first collision, the incident particle is not moving. The leading particle of the clump is moving with the incident velocity while the rest of the clump is unaffected.

Suppose that a single point particle collides with a springy clump that happens to have a leading particle with the same mass as the incident particle. We know from the collision matrix that immediately after the first collision, the incident particle will be motionless while the leading particle of the clump will move off with the incident particle's velocity. The rest of the clump will not yet have been affected because no displacement of the springs has occurred: the position of the leading particle hasn't changed, only its velocity has changed. The aftermath of the first collision is shown in figures [3.2] and [3.3].

If this is the only collision that occurs, we can calculate the coefficient of restitution directly from the conservation of momentum. After the first collision, the clump must have the velocity given in equation [3.2] to satisfy conservation of momentum.[1]

$$V = \frac{mv}{M} \qquad [3.2]$$

If there are no subsequent collisions, the coefficient of restitution is thus

$$e = \frac{m}{M} \qquad [3.3]$$

If the clump consisted of equally-spaced particles of equal mass, we might consider it to be a model of a cylinder N times larger than the point particle and equation [3.3] would be in agreement with the cylinder collision formula from chapter two.

Unfortunately, it is not obvious whether there will be a second collision. Even if we know that there will be a second collision, it would be much more complicated to analyze because the clump would be internally vibrating when it returns to hit the incident point particle. What we need is a description of how springy clumps that have internal energy move so that we can determine whether the second collision will occur, and what effect it might have.

Let us begin with a springy clump consisting of two point particles and a single spring. The Lagrangian for this system is given by equation [3.4].

---

[1] Capitalized quantities are for the clump while lowercase quantities are for the incident particle.

$$L = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 - \frac{1}{2}k\left(x_2 - x_1 - a\right)^2 \qquad [3.4]$$

The variables x_1 and x_2 are the positions of the particles in the clump and a is the equilibrium separation distance (where the spring exerts no force). The Lagrangian has two basic parts: The first two terms are the kinetic energy contribution, and the last term is the spring interaction term. If the last term were removed or the spring constant were set to zero, we would have the Lagrangian for two free particles.

We expect the center of mass velocity of the clump to be a conserved quantity, and we can change variables in the Lagrangian to make this conservation manifest. In fact, we would like to transform to the centralized basis that was so important in the first chapter, except this time we need to consider the positions of the particles as well as their velocities so that we can find the spring's contribution. The new variables are given in equations [3.5].

$$X \equiv \frac{m_1 x_1 + m_2 x_2}{M},\ \chi \equiv x_2 - x_1 \qquad [3.5]$$

Notice that the derivatives of the positions in equations [3.5] reproduce our earlier definitions of the center of mass velocity and the relative velocity (equation [3.6]).

$$\begin{pmatrix} \dot{X} \\ \dot{\chi} \end{pmatrix} = \begin{pmatrix} \dfrac{m_1 v_1 + m_2 v_2}{M} \\ v_2 - v_1 \end{pmatrix} = \begin{pmatrix} V \\ \upsilon \end{pmatrix} \qquad [3.6]$$

The inverses of these definitions are

$$x_1 = X - \frac{m_2}{M}\chi,\ x_2 = X + \frac{m_1}{M}\chi \qquad [3.7]$$

Using the new variables, we can rewrite the Lagrangian for the two-mass clump.

$$L = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}\mu\dot{\chi}^2 - \frac{1}{2}k\left(\chi - a\right)^2 \qquad [3.8]$$

The usual mass definitions are used.

$$M \equiv m_1 + m_2,\ \mu \equiv \frac{m_1 m_2}{m_1 + m_2} \qquad [3.9]$$

The Euler-Lagrange equations of motion are calculated from equation [3.8].

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{X}}\right) = M\ddot{X} = 0 = \frac{\partial L}{\partial X},\ \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{\chi}}\right) = \mu\ddot{\chi} = -k\left(\chi - a\right) = \frac{\partial L}{\partial \chi} \qquad [3.10]$$

Equations [3.10] are familiar as the conservation of momentum of the center of mass and the force equation for the simple harmonic oscillator, respectively. The solutions are given in equations [3.11].

$$X(t) = \dot{X}_0 t + X_0,\ \chi(t) = A\sin(\omega t + \delta) + a,\ \omega = \sqrt{\frac{k}{\mu}} \qquad [3.11]$$

For the collision of a point particle into the two-particle clump, figure [3.3] gives the constants of the motion after the first collision. The initial positions and velocities of the particles are

$$x_1 = 0,\ \dot{x}_1 = v,\ x_2 = a,\ \dot{x}_2 = 0 \qquad [3.12]$$

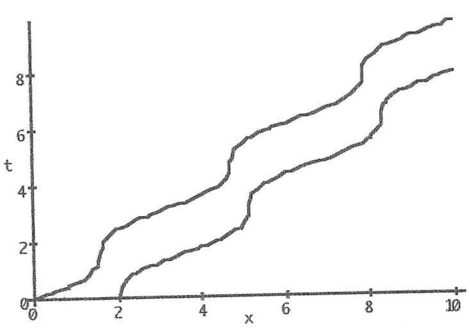
Equations [3.5] determine the initial values of the centralized positions and velocities (setting $x = 0$ and $t = 0$ at the point of collision). When these initial values are used in the equations of motion the results are

$$X(t) = \frac{m_1}{M}vt + \frac{m_2}{M}a,\ \chi(t) = -\frac{v}{\omega}\sin(\omega t) + a \qquad [3.13]$$

This gives the equations of motion for the particles in the clump as

$$x_1 = \frac{m_1}{M}vt + \frac{m_2}{M}\frac{v}{\omega}\sin(\omega t), \qquad [3.14]$$

$$x_2 = \frac{m_1}{M}\left(vt - \frac{v}{\omega}\sin(\omega t)\right) + a$$



[3.4] -- The behavior of a two-particle clump ($m_1 = m_2$, $\omega = 2$, $v = 2$) after being struck as in figures [3.2] and [3.3].

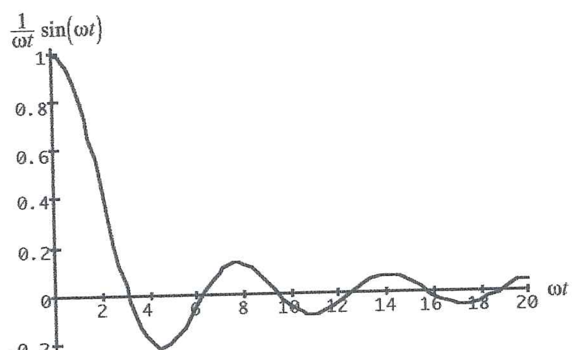
An example of a space-time diagram of this sort of motion is shown in figure [3.4].

We are now able to find out whether (and when) a second collision occurs with a particular collection of masses, spring constants, and velocities. If a second collision were to occur, the left particle of the clump would have to return to where the incident point particle ended up after the first collision.

$$0 = \frac{m_1}{M}vt + \frac{m_2}{M}\frac{v}{\omega}\sin(\omega t) \qquad [3.15]$$

Equation [3.15] simplifies to equation [3.16].

$$-\frac{m_1}{m_2} = \frac{1}{\omega t}\sin(\omega t) \qquad [3.16]$$



[3.5] -- This is a plot of the right-hand side of equation [3.16].

The right-hand side of equation [3.16] is plotted in figure [3.5].

Figure [3.5] makes it clear that the second collision will only occur for a very limited range of masses: unless m_2 is greater than about five times m_1 the second collision will not occur.

If the masses are equal, we have only one collision, and our result agrees with the cylinder collision formula! In addition, we know that as long as m_1 is not too small compared to m_2 there will be only one collision between a point particle and a two-particle springy clump, in which case the coefficient of restitution is given by [3.3].

What about an N-particle clump? How does it move? We need to generalize the results of the two-particle clump.

We would like a Lagrangian that takes care of the N free particle terms and the N-1 interaction terms. Equation [3.17] does just this.

$$L=\frac{1}{2}\sum_{i=1}^{N} m_i\dot{x}_i^2-\frac{1}{2}\sum_{j=1}^{N-1} k_{j,j+1}\left(x_{j+1}-x_j-\left(x_{j+1}^{equil.}-x_j^{equil.}\right)\right)^2 \qquad [3.17]$$

The equilibrium position of the j^th particle is $x_j^{equil.}$.

The Lagrangian in equation [1.17] is messy, however. It has the same rough form as the energy equations from chapter one, so we can use matrices and vectors in much the same way.

$$L=\frac{1}{2}\dot{\mathbb{x}}^{\dashv}\,\mathbb{M}\,\dot{\mathbb{x}}-\frac{1}{2}\mathbb{x}^{\dashv}\,\mathbb{K}\,\mathbb{x} \qquad [3.18]$$

Where

$$\mathbb{M}=Diag\left(m_1, m_2, \ldots, m_N\right),\ \mathbb{x}=\begin{pmatrix} x_1-x_1^{equil.} \\ x_2-x_2^{equil.} \\ \vdots \\ x_N-x_N^{equil.} \end{pmatrix},\ \dot{\mathbb{x}}=\frac{d\mathbb{x}}{dt}=\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_N \end{pmatrix} \qquad [3.19]$$

$\mathbb{K}$ is more difficult to determine. We express the interaction term as the product of a matrix and its transpose.

$$\mathbb{x}^{\dashv}\mathbb{K}\mathbb{x}=\left(\sqrt{\mathbb{K}}\mathbb{x}\right)^{\dashv}\left(\sqrt{\mathbb{K}}\mathbb{x}\right),\ \sqrt{\mathbb{K}}=\begin{pmatrix} k_{1,2}^{-1/2} & -k_{1,2}^{-1/2} & & & \mathbf{0} \\ & k_{2,3}^{-1/2} & -k_{2,3}^{-1/2} & & \\ & & \ddots & \ddots & \\ & & & k_{N-1,N}^{-1/2} & -k_{N-1,N}^{-1/2} \\ \mathbf{0} & & & & 0 \end{pmatrix} \qquad [3.20]$$

Multiplying the transpose of $\sqrt{\mathbb{K}}$ and $\sqrt{\mathbb{K}}$, we obtain $\mathbb{K}$.

$$\mathbb{K} = \sqrt{\mathbb{K}}^{\dashv}\sqrt{\mathbb{K}} = \begin{pmatrix} k_{1,2} & -k_{1,2} & 0 & & \mathbf{0} \\ -k_{1,2} & k_{1,2}+k_{2,3} & -k_{2,3} & & \\ & \ddots & \ddots & \ddots & \\ & -k_{N-2,N-1} & k_{N-2,N-1}+k_{N-1,N} & -k_{N-1,N} & \\ \mathbf{0} & 0 & -k_{N-1,N} & k_{N-1,N} & \end{pmatrix} \qquad [3.21]$$

Despite the complexity of the matrices that make up the Lagrangian, we have now found the absolutely general Lagrangian for particles on springs.

Now, the Euler-Lagrange equations of motion that arise from the Lagrangian in equation [3.18] are all in the form of equation [3.22].

$$m_i\ddot{x}_i = -k_{i-1,i}\left(x_i - x_{i-1} - \left(x_i^{equil.} - x_{i-1}^{equil.}\right)\right) + k_{i,i+1}\left(x_{i+1} - x_i - \left(x_{i+1}^{equil.} - x_i^{equil.}\right)\right) \qquad [3.22]$$

The Euler-Lagrange equations are N coupled ordinary differential equations. All N of these equations are contained in the matrix equation [3.23].

$$\mathbb{M}\,\ddot{\mathbb{x}} = -\mathbb{K}\,\mathbb{x} \qquad [3.23]$$

Equation [3.23] is precisely what we might have guessed by looking at the Lagrangian.

The mass matrix is diagonal, so we immediately know its inverse.

$$\ddot{\mathbb{x}} = -\mathbb{M}^{-1}\,\mathbb{K}\,\mathbb{x} \qquad [3.24]$$

$$\mathbb{M}^{-1} = Diag\left(m_1^{-1}, m_2^{-1}, \dots, m_N^{-1}\right) \qquad [3.25]$$

Now, all we need to do is solve equation [3.24] and we will know how any springy clump behaves without external forces. If $\mathbb{M}^{-1}\,\mathbb{K}$ has no repeated eigenvalues, then the matrix $\mathbb{D}$ constructed using the eigenvectors as columns diagonalizes $\mathbb{M}^{-1}\,\mathbb{K}$.

$$\mathbb{D}^{-1}\ddot{\mathbb{x}} = -\mathbb{D}^{-1}\,\mathbb{M}^{-1}\,\mathbb{K}\,\mathbb{D}\,\mathbb{D}^{-1}\,\mathbb{x} \qquad [3.26]$$

The matrix on the right is a diagonal matrix, the **frequency matrix**.

$$\mathbb{D}^{-1}\,\mathbb{M}^{-1}\,\mathbb{K}\,\mathbb{D} = Diag\left(\lambda_1, \lambda_2, \dots, \lambda_N\right) \qquad [3.27]$$

If we write the solutions as the real part of a complex equation, the phase of the oscillations is automatically determined from the initial values, because the velocity, as the complex part, fixes the phase (equation [3.28]).

$$\left[\mathbb{D}^{-1}\mathfrak{x}\right]_j = \mathrm{Re}\left[\left[\left[\mathbb{D}^{-1}\mathfrak{x}^{initial}\right]_j - \frac{i}{\sqrt{\lambda_j}}\left[\mathbb{D}^{-1}\dot{\mathfrak{x}}^{initial}\right]_j\right]e^{it\sqrt{\lambda_j}}\right] \qquad [3.28]$$

Equation [3.28] applies if the eigenvalue is non-zero, but we know from equation [3.24] that the eigenvalue corresponding to the center of mass motion will be zero. If the eigenvalue is zero, then our solution is given by equation [3.29].

$$\left[\mathbb{D}^{-1}\mathfrak{x}\right]_j = \left[\mathbb{D}^{-1}\dot{\mathfrak{x}}^{initial}\right]_j t + \left[\mathbb{D}^{-1}\mathfrak{x}^{initial}\right]_j \qquad [3.29]$$

Exactly one of the eigenvalues will be zero, the one associated with the center of mass, if we choose this to be the first coordinate, then we can write

$$\mathfrak{x} = \mathrm{Re}\left[\mathbb{D}\,\mathbb{A}\left(\mathbb{D}^{-1}\mathfrak{x}^{initial} - i\,\mathbb{\omega}^{-1}\mathbb{D}^{-1}\dot{\mathfrak{x}}^{initial}\right)\right] \qquad [3.30]$$

All the time dependence in equation [3.30] is contained in the $\mathbb{A}$ matrix.

$$\mathbb{A} = Diag\left(1 + i\varpi t,\, e^{i\omega_2 t},\, e^{i\omega_3 t},\, \dots,\, e^{i\omega_N t}\right),\; \mathbb{\omega}^{-1} = Diag\left(\frac{1}{\varpi}, \frac{1}{\sqrt{\lambda_2}}, \frac{1}{\sqrt{\lambda_3}}, \dots, \frac{1}{\sqrt{\lambda_N}}\right) \qquad [3.31]$$

The quantity ϖ found in the center of mass component is an arbitrary constant that immediately cancels out but is needed so that the $\mathbb{A}$ matrix and the frequency matrix have components with consistent units.

The particles' positions are given by equation [3.32].

$$\mathbb{x} = \mathrm{Re}\left[\mathbb{D}\,\mathbb{A}\left(\mathbb{D}^{-1}\mathfrak{x}^{initial} - i\,\mathbb{\omega}^{-1}\mathbb{D}^{-1}\dot{\mathfrak{x}}^{initial}\right)\right] + \mathbb{x}_0 \qquad [3.32]$$

Where

$$\mathbb{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix},\; \mathbb{x}_0 = \begin{pmatrix} x_1^{equil.} \\ x_2^{equil.} \\ \vdots \\ x_N^{equil.} \end{pmatrix} \qquad [3.33]$$

The vector $\mathbb{x}_0$ contains the equilibrium positions. It cannot contain any of the "stretching" of the springs because it is not varied by the $\mathbb{A}$ matrix.

Equation [3.32] describes how a springy clump behaves when there are no external influences.

Equation [3.32] consists of four parts. The first part consists of the three vectors $\mathbb{x}_0$, $\mathfrak{x}$, and $\dot{\mathfrak{x}}$. They determine the state of the clump. By state, I mean the amplitudes and phase of the oscillations. Unfortunately, these initial values are given in terms of the particles' positions and velocities, and the particles do not oscillate independently. If one particle in a stationary clump is wiggled, the other particles do not remain stationary. The oscillations occur in a different representation where the frequency matrix is diagonalized.

This brings us to the second part of equation [3.32]. The $\mathbb{D}^{-1}$ matrices convert the initial value vectors based on particle position to a new basis. In this basis, the components are the eigenvectors of the frequency matrix. This means that each oscillation, or normal mode, has its own amplitude, phase, and frequency. The amplitude and phase are derived from linear combinations of the components of the initial value vectors, and the frequency is the square root of the eigenvalue correlated with the normal mode. Center of mass motion is also a normal mode. The normal modes are independent: if a clump is vibrating purely in one normal mode, it will remain in that normal mode until an outside force changes its state. All motions of the clump when it is free from outside interference are linear combinations of normal modes.

The vibrations occur in the eigenspace, that is, the time dependent portion of equation [3.32], the $\mathbb{A}$ matrix, acts differently on each normal mode. The normal mode evolves with time because of the time dependence of $\mathbb{A}$. The $\mathbb{A}$ matrix is the third part of equation [3.32].

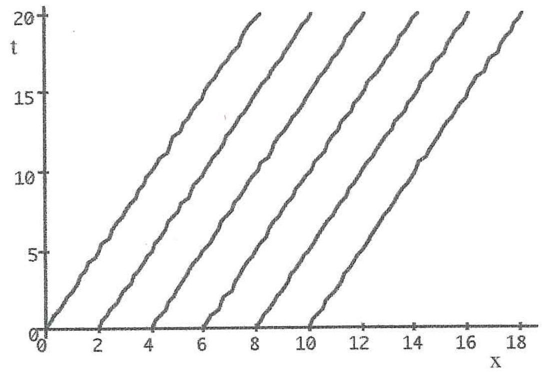
The final part of equation [3.32] is the $\mathbb{D}$ matrix, which converts back from the eigenspace to the positions of the particles.

As an example, the normal modes of a six-particle clump (with equal spring constants, particle masses and energies) are shown in figures [3.6] to [3.11] in order of increasing frequency.

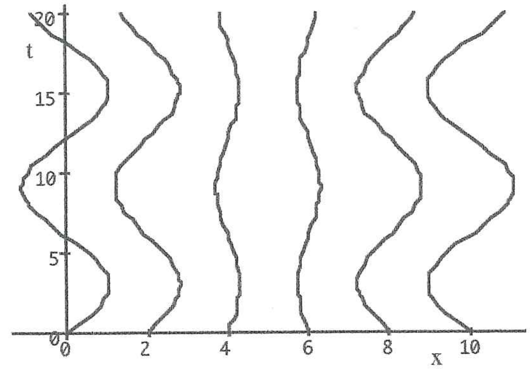
We can now return to the collision of a point particle with an N-particle springy cylinder. When we set all of the particle masses equal and all of the spring constants equal, we find that the matrix we need to diagonalize is

$$\mathbb{M}^{-1}\mathbb{K} = \frac{k}{m}\begin{pmatrix} 1 & -1 & & & 0 \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ 0 & & & -1 & 1 \end{pmatrix} \qquad [3.34]$$

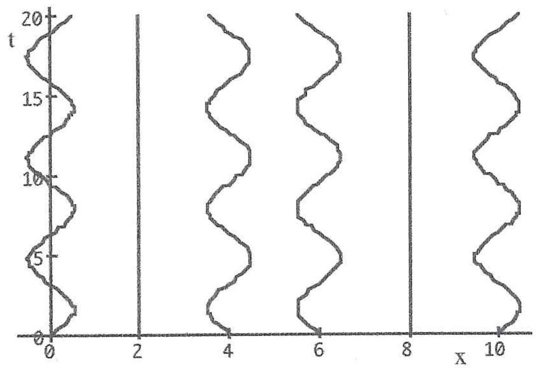
The diagonalizing matrices are given for the first five cases in table [3.12].



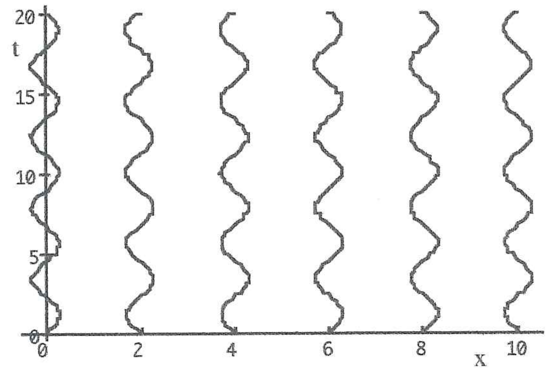
[3.6] -- Center of mass motion is the first normal mode of the six-particle clump.



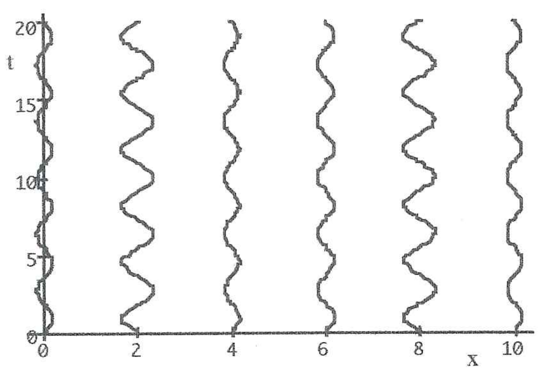
[3.7] -- All the particles move in and out in concert in the second normal mode. The second normal mode has an angular frequency of $\sqrt{(2-\sqrt{3})\frac{k}{m}}$.



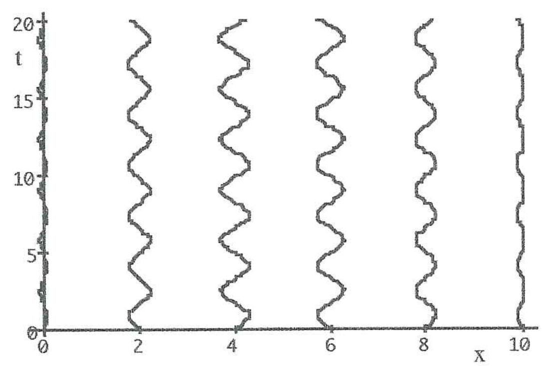
[3.8] -- The third normal mode of the six-particle clump has an angular frequency of $\sqrt{\frac{k}{m}}$. Two of the particles are stationary in this normal mode.



[3.9] -- The fourth normal mode of the six-particle clump has an angular frequency of $\sqrt{2\frac{k}{m}}$.



[3.10] -- The fifth normal mode of the six-particle clump has an angular frequency of $\sqrt{3\frac{k}{m}}$.



[3.11] -- The sixth and final normal mode of the six-particle clump has an angular frequency of $\sqrt{(2+\sqrt{3})\frac{k}{m}}$.

$\mathbb{D}$	Eigenvalues of the frequency matrix
$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$	$\lambda_1 = 0$ $\lambda_2 = 2\frac{k}{m}$
$\frac{1}{\sqrt{3}}\begin{pmatrix} 1 & \sqrt{3/2} & \sqrt{1/2} \\ 1 & 0 & -\sqrt{2} \\ 1 & -\sqrt{3/2} & \sqrt{1/2} \end{pmatrix}$	$\lambda_1 = 0$ $\lambda_2 = \frac{k}{m}$ $\lambda_3 = 3\frac{k}{m}$
$\frac{1}{\sqrt{4}}\begin{pmatrix} 1 & \frac{1}{\sqrt{Q}} & 1 & \frac{1}{\sqrt{4-Q}} \\ 1 & \frac{1-Q}{\sqrt{Q}} & -1 & \frac{1-[4-Q]}{\sqrt{4-Q}} \\ 1 & \frac{Q-1}{\sqrt{Q}} & -1 & \frac{[4-Q]-1}{\sqrt{4-Q}} \\ 1 & \frac{-1}{\sqrt{Q}} & 1 & \frac{-1}{\sqrt{4-Q}} \end{pmatrix}$	$\lambda_1 = 0$ $\lambda_2 = Q\frac{k}{m}$ $\lambda_3 = 2\frac{k}{m}$ $\lambda_4 = [4-Q]\frac{k}{m}$ $Q \equiv (2-\sqrt{2})$
$\frac{1}{\sqrt{5}}\begin{pmatrix} 1 & \frac{\sqrt{5}}{\sqrt{2(2-G)}} & \frac{1}{\sqrt{2(1-G)}} & \frac{\sqrt{5}}{\sqrt{2(3+G)}} & \frac{1}{\sqrt{2(2+G)}} \\ 1 & \frac{\sqrt{5}G}{\sqrt{2(2-G)}} & \frac{G-1}{\sqrt{2(1-G)}} & \frac{-\sqrt{5}(G+1)}{\sqrt{2(3+G)}} & \frac{-G-2}{\sqrt{2(2+G)}} \\ 1 & 0 & \frac{-2G}{\sqrt{2(1-G)}} & 0 & \frac{2G+2}{\sqrt{2(2+G)}} \\ 1 & \frac{-\sqrt{5}G}{\sqrt{2(2-G)}} & \frac{G-1}{\sqrt{2(1-G)}} & \frac{\sqrt{5}(G+1)}{\sqrt{2(3+G)}} & \frac{-G-2}{\sqrt{2(2+G)}} \\ 1 & \frac{-\sqrt{5}}{\sqrt{2(2-G)}} & \frac{1}{\sqrt{2(1-G)}} & \frac{-\sqrt{5}}{\sqrt{2(3+G)}} & \frac{1}{\sqrt{2(2+G)}} \end{pmatrix}$	$\lambda_1 = 0$ $\lambda_2 = [1-G]\frac{k}{m}$ $\lambda_3 = [2-G]\frac{k}{m}$ $\lambda_4 = [2+G]\frac{k}{m}$ $\lambda_5 = [3+G]\frac{k}{m}$ $G \equiv \frac{\sqrt{5}-1}{2}$
$\frac{1}{\sqrt{6}}\begin{pmatrix} 1 & \frac{1}{\sqrt{2R}} & \sqrt{\frac{3}{2}} & 1 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2[4-R]}} \\ 1 & \frac{1-R}{\sqrt{2R}} & 0 & -1 & -\frac{2}{\sqrt{2}} & \frac{1-[4-R]}{\sqrt{2[4-R]}} \\ 1 & \frac{R}{\sqrt{2R}} & -\sqrt{\frac{3}{2}} & -1 & \frac{1}{\sqrt{2}} & \frac{[4-R]}{\sqrt{2[4-R]}} \\ 1 & \frac{-R}{\sqrt{2R}} & -\sqrt{\frac{3}{2}} & 1 & \frac{1}{\sqrt{2}} & \frac{-[4-R]}{\sqrt{2[4-R]}} \\ 1 & \frac{R-1}{\sqrt{2R}} & 0 & 1 & -\frac{2}{\sqrt{2}} & \frac{[4-R]-1}{\sqrt{2(4-R)}} \\ 1 & \frac{-1}{\sqrt{2R}} & \sqrt{\frac{3}{2}} & -1 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2(4-R)}} \end{pmatrix}$	$\lambda_1 = 0$ $\lambda_2 = R\frac{k}{m}$ $\lambda_3 = \frac{k}{m}$ $\lambda_4 = 2\frac{k}{m}$ $\lambda_5 = 3\frac{k}{m}$ $\lambda_6 = [4-R]\frac{k}{m}$ $R \equiv 2-\sqrt{3}$

[3.12] -- The transformation matrices that cause $\mathbb{D}^{-1}\mathbb{M}^{-1}\mathbb{K}\,\mathbb{D}$ to be diagonal (and thus have the eigenvectors of $\mathbb{M}^{-1}\mathbb{K}$ as columns) and the corresponding eigenvalues for the first five springy clumps with equal masses and equal spring constants.

With the diagonalizing matrices we can return to our problem of colliding a point particle into a clump. We have four more clumps in addition to the two-particle case already solved.

The procedure is simple. We have a set of initial conditions that we know from figure [3.3]. That is, all the particles in the clump are at their equilibrium positions, and only the first particle is moving. We construct the initial value vectors:

$$\mathbb{X}_0 = \begin{pmatrix} 0 \\ a \\ 2a \\ \vdots \\ Na \end{pmatrix}, \quad \mathfrak{X}^{initial} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \dot{\mathfrak{X}}^{initial} = \begin{pmatrix} v \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad [3.35]$$

Now, we apply equation [3.32], and we have our solutions for the motion of the clumps. For all the cylinders, $\mathbb{D}$ is unitary, because I have normalized the eigenvectors before using them as columns of the matrix. This makes our job even easier because $\mathbb{D}^{-1} = \mathbb{D}^{\dashv}$.

When we apply $\mathbb{D}^{-1}$ to our initial conditions, we find the following motions for the leading particle of the clump (after the first collision).

$$x_1 = \frac{v\,t}{2} + \frac{v\sin\left(\sqrt{2}\omega t\right)}{2\sqrt{2}\omega} \qquad [3.37]$$

$$x_1 = \frac{v\,t}{3} + \frac{v\sin\left(\omega t\right)}{2\omega} + \frac{v\sin\left(\sqrt{3}\,\omega t\right)}{6\sqrt{3}\,\omega}$$

$$x_1 = \frac{v\,t}{4} + \frac{v\sin\left(\sqrt{Q}\omega t\right)}{4Q^{3/2}\omega} + \frac{v\sin\left(\sqrt{2}\omega t\right)}{4\sqrt{2}\omega} + \frac{v\sin\left(\sqrt{4-Q}\,\omega\, t\right)}{4[4-Q]^{3/2}\omega}$$

$$x_1 = \frac{v\,t}{5} + \frac{v\sin\left(\sqrt{1-G}\omega\, t\right)}{2[2-G]\sqrt{1-G}\omega} + \frac{v\sin\left(\sqrt{2-G}\omega\, t\right)}{10[1-G]\sqrt{2-G}\omega} + \frac{v\sin\left(\sqrt{2+G}\omega\, t\right)}{2[3+G]\sqrt{2+G}\omega} + \frac{v\sin\left(\sqrt{3+G}\omega\, t\right)}{10[2+G]\sqrt{3+G}\omega}$$

$$x_1 = \frac{v\,t}{6} + \frac{v\sin\left(\sqrt{R}\omega\, t\right)}{12R^{3/2}\,\omega} + \frac{v\sin\left(\omega\, t\right)}{4\omega} + \frac{v\sin\left(\sqrt{2}\omega\, t\right)}{6\sqrt{2}\omega} + \frac{v\sin\left(\sqrt{3}\omega\, t\right)}{12\sqrt{3}\,\omega} + \frac{v\sin\left(\sqrt{[4-R]}\,\omega\, t\right)}{12[4-R]^{3/2}\,\omega}$$

$$\omega \equiv \sqrt{\frac{k}{m}} \qquad [3.38]$$

The motions of all the particles in the clumps are depicted in the space-time diagrams of figures [3.13] through [3.17].

In all the figures, where the frequencies and velocities were chosen arbitrarily, the leading particle does not return to collide with the point particle that is stationary at x=0, but is this true in general? Looking back at the positions of the leading particles in equations [3.37], we can see that if the leading particle were to return back to hit the incident point particle, it would have to be the case that

$$0 = \frac{vt}{N} + \sum_{n=2}^{N} \frac{a_n}{b_n\omega} v \sin(b_n\omega t) \qquad [3.39]$$

Because initially only the leading particle was moving, we can say that the derivative of x_1 began at v, so

$$v = \frac{v}{N} + \sum_{n=2}^{N} a_n v \cos(0) \qquad [3.40]$$

Or

$$\frac{N-1}{N} = \sum_{n=2}^{N} a_n \qquad [3.41]$$

Now we can perform a Taylor expansion of equation [3.39].

$$0 = \frac{t}{N} + \sum_{n=2}^{N} \frac{a_n}{b_n\omega} \sum_{p=0}^{\infty} \frac{(-1)^p}{(2p+1)!} (b_n\omega t)^{2p+1} \qquad [3.42]$$

Or

$$-\frac{1}{N} = \sum_{n=2}^{N} a_n - \sum_{n=2}^{N} a_n \frac{(b_n\omega t)^2}{3!} + \sum_{n=2}^{N} a_n \frac{(b_n\omega t)^4}{5!} - \ldots \qquad [3.43]$$

Finally

$$0 = 1 - \sum_{n=2}^{N} a_n \frac{(b_n\omega t)^2}{3!} + \sum_{n=2}^{N} a_n \frac{(b_n\omega t)^4}{5!} - \ldots \qquad [3.44]$$

For this to hold, the second order term must dominate the fourth order term by almost one. The only way for this to happen is if ω is tiny. Because the time interval we are concerned with is of roughly the same order as the frequency, this is, to a fair approximation, equivalent to the quadratic

$$0 = 5! - 5 \cdot 4 \sum_{n=2}^{N} a_n (b_n\omega t)^2 + \sum_{n=2}^{N} a_n (b_n\omega t)^4 \qquad [3.45]$$

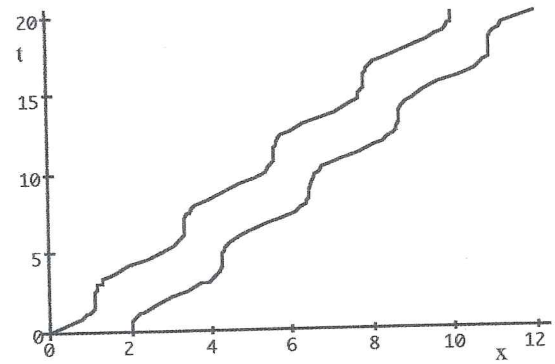
Which has real solutions only if

$$\left(5 \cdot 4 \sum_{n=2}^{N} a_n\right)^2 \geq 4 \cdot 5! \sum_{n=2}^{N} a_n \qquad [3.46]$$

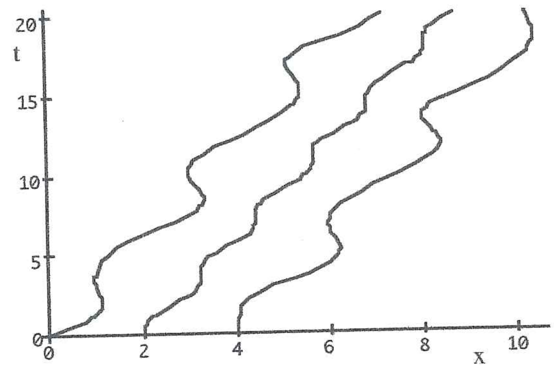
However, using equation [3.41] again, this is equation [3.47].

$$\frac{N-1}{N} \geq \frac{6}{5} \qquad [3.47]$$

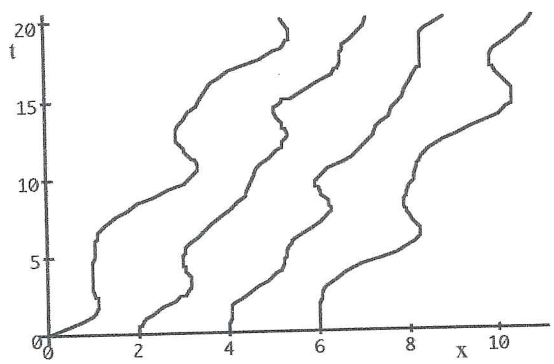
The relationship described by equation [3.47] obviously cannot occur. Thus, we presume that unless the spring constant is small in comparison to the mass, there is only one collision when a point particle hits a cylinder-like springy clump.



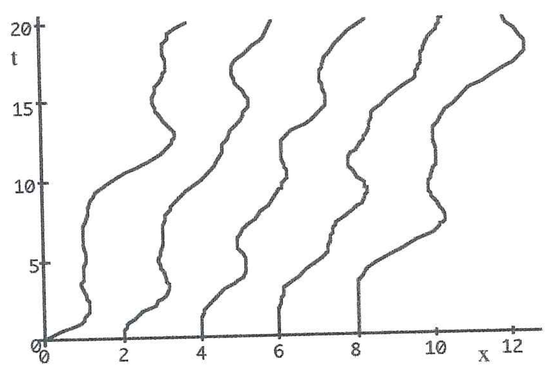
[3.13] -- The two-particle clump with equal masses and equal string constants.



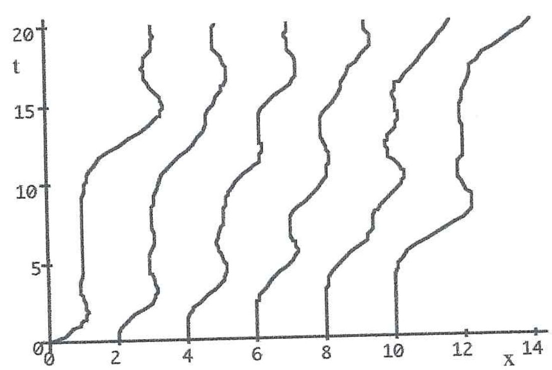
[3.14] -- The three-particle clump with equal masses and equal string constants.



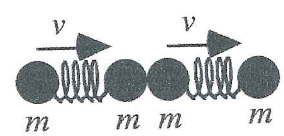
[3.15] -- The four-particle clump with equal masses and equal string constants.



[3.16] -- The five-particle clump with equal masses and equal string constants.



[3.17] -- The six-particle clump with equal masses and equal string constants.



[3.18] -- Immediately after a two-particle clump approaching from the left hits a stationary two-particle clump, the left-hand particles in each clump are moving and the right-hand particles are not.

As I argued earlier, because there is only one collision the coefficient of restitution must be $1/N$, and we have agreement with the cylinder collision formula. The preceding figures are the motions of the particles in springy clumps after they are hit by a point particle.[2]

We have agreement with the cylinder collision formula in the case of a point particle hitting a clump, but what about clumps hitting clumps? The cylinder collision formula predicts that equal length cylinders collide elastically, this will be the case first analyzed.

Immediately after a two-particle cylinder hits a two-particle cylinder, the situation is the one in figure [3.18]. The left-hand particles in each clump are moving while the right-hand particles are not. We have already solved for this kind of motion! After the first collision, the clumps behave just as the clumps that were struck on the left by a point particle behaved.

We know that the motion of the left particles will be[3]

$$x_1^L + a = \frac{v\,t}{2} + \frac{v\sin\left(\sqrt{2}\omega t\right)}{2\sqrt{2}\omega} = x_1^R \qquad [3.48]$$

The right particles' motion will be

$$x_2^L = \frac{v\,t}{2} - \frac{v\sin\left(\sqrt{2}\omega t\right)}{2\sqrt{2}\omega} = x_2^R - a \qquad [3.49]$$

The collisions occur when

$$x_2^L = \frac{v\,t}{2} - \frac{v\sin\left(\sqrt{2}\omega t\right)}{2\sqrt{2}\omega} = \frac{v\,t}{2} + \frac{v\sin\left(\sqrt{2}\omega t\right)}{2\sqrt{2}\omega} = x_1^R \qquad [3.50]$$

Or

$$\sin\left(\sqrt{2}\omega t\right) = 0 \qquad [3.51]$$

Equation [3.51] accurately reproduces the collision at $t = 0$, but it also predicts a second collision that occurs when $\sqrt{2}\omega t = \pi$.

We need to know the velocities of the particles when this second collision occurs to calculate the subsequent motion. Taking the derivative of equations [3.48] and [3.49] we find that at the time of the second collision,

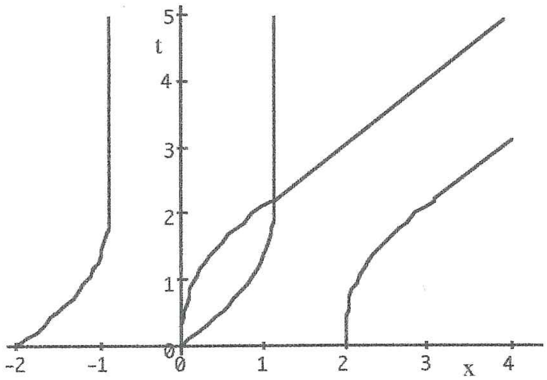
$$\dot{x}_1^L\left(t = \pi/\sqrt{2}\omega\right) = \dot{x}_1^R\left(t = \pi/\sqrt{2}\omega\right) = 0,\ \ \dot{x}_2^L\left(t = \pi/\sqrt{2}\omega\right) = \dot{x}_2^R\left(t = \pi/\sqrt{2}\omega\right) = v \qquad [3.52]$$

---

[2]In the figures, the constants are chosen so that the incident velocity is one length unit per time unit, and the spring constant divided by the mass is one time unit squared, i.e., $\omega = v = 1$.

[3]The superscript L or R determines where the particle is: the left clump or the right clump. The subscript distinguishes the left particle from the right particle within a clump.

The second collision occurs between the x_2^L particle moving to the right with speed v and the x_1^R particle that is not moving. After this second collision, both particles in the right clump are moving to the right with speed v while the left clump is not moving at all! The collision is elastic, no internal energy remains, and the cylinder collision formula is again correct. A space-time diagram of the entire affair is shown in figure [3.19].



[3.19] -- The collision of two two-particle clumps. This collision is elastic.

This collision is simpler than the other collisions of a clump into a clump because there are fewer particles. The next most simple collision of clumps with equal lengths is the collision of three-particle clumps. This time, to further simplify things, we will consider the collision in the center-of-mass frame where the collision is symmetrical about $x = 0$. Again, we begin immediately after the first collision of the leading particles. The initial position and velocity vectors are given in equations [3.53].

$$\mathbb{x}^L(0) = \begin{pmatrix} -2a \\ -a \\ 0 \end{pmatrix},\ \dot{\mathbb{x}}^L(0) = \begin{pmatrix} v \\ v \\ -v \end{pmatrix},\ \mathbb{x}^R(0) = \begin{pmatrix} 0 \\ a \\ 2a \end{pmatrix},\ \dot{\mathbb{x}}^R(0) = \begin{pmatrix} v \\ -v \\ -v \end{pmatrix} \qquad [3.53]$$

After the first collision, the motions of the particles in the left clump are

$$\mathbb{x}^L(t) = \begin{pmatrix} \frac{v\,t}{3} + \frac{v\sin(\omega t)}{\omega} - \frac{v\sin(\sqrt{3}\,\omega t)}{3\sqrt{3}\,\omega} - 2a \\ \frac{v\,t}{3} + \frac{v\sin(\sqrt{3}\,\omega t)}{3\sqrt{3}\,\omega} - a \\ \frac{v\,t}{3} - \frac{v\sin(\omega t)}{\omega} - \frac{v\sin(\sqrt{3}\,\omega t)}{3\sqrt{3}\,\omega} \end{pmatrix} \qquad [3.54]$$

And, as mirror symmetry suggests, the right clump's particles are located by

$$\mathbb{x}^R(t) = \begin{pmatrix} -\frac{v\,t}{3} + \frac{v\sin(\omega t)}{\omega} + \frac{v\sin(\sqrt{3}\,\omega t)}{3\sqrt{3}\,\omega} \\ -\frac{v\,t}{3} - \frac{2\,v\sin(\sqrt{3}\,\omega t)}{3\sqrt{3}\,\omega} + a \\ -\frac{v\,t}{3} - \frac{v\sin(\omega t)}{\omega} + \frac{v\sin(\sqrt{3}\,\omega t)}{3\sqrt{3}\,\omega} + 2a \end{pmatrix} \qquad [3.55]$$

A second collision occurs when the x_1^R particle and the x_3^L particle return to the origin. This second collision happens at time t_{c2}, given implicitly by

$$\frac{v\,t_{c2}}{3} - \frac{v\sin(\omega\,t_{c2})}{\omega} - \frac{v\sin(\sqrt{3}\,\omega\,t_{c2})}{3\sqrt{3}\,\omega} = -\frac{v\,t_{c2}}{3} + \frac{v\sin(\omega\,t_{c2})}{\omega} + \frac{v\sin(\sqrt{3}\,\omega\,t_{c2})}{3\sqrt{3}\,\omega} \qquad [3.56]$$

Or

$$\sqrt{3}\omega\;t_{c2} - 3\sqrt{3}\sin(\omega\,t_{c2}) - \sin(\sqrt{3}\,\omega\,t_{c2}) = 0 \qquad [3.57]$$

This equation is difficult to solve, but for our purposes, a numerical solution will suffice.[4]

$$t_{c2} = 2.162382884\frac{1}{\omega} \qquad [3.58]$$

At this time, the particles have the following velocities

$$\dot{\mathbf{x}}^L(t_{c2}) = \begin{pmatrix} 0.050056093v \\ -2.154695004v \\ 1.165413407v \end{pmatrix},\; \dot{\mathbf{x}}^R(t_{c2}) = \begin{pmatrix} -1.165413407v \\ 2.154695004v \\ -0.050056093v \end{pmatrix} \qquad [3.59]$$

The leading particles collide. They have equal masses, so their velocities are just swapped after the collision.

$$\dot{x}_3^L(t+\varepsilon) = -1.165413407v,\; \dot{x}_1^R(t+\varepsilon) = 1.165413407v \qquad [3.60]$$

That is not quite the end of the story, if the clumps are to collide elastically, there will be one more collision where all the internal energy will be converted to translational energy. This last collision is more complicated, however, because the particles have initial displacements and so are mid-oscillation when they collide. The leading particles' positions after the second collision are given by

$$\begin{aligned} x_3^L = {} & 0.7207942948\tfrac{v}{\omega} - 0.4436089381\tfrac{v}{\omega}(t - t_{c2}) \\ & + 0.6077347500\tfrac{v}{\omega}\sin\left[\omega(t - t_{c2})\right] - 0.8300569348\tfrac{v}{\omega}\cos\left[\omega(t - t_{c2})\right] \\ & + 0.06585818291\tfrac{v}{\omega}\sin\left[\omega\sqrt{3}\,(t - t_{c2})\right] + 0.1092626404\tfrac{v}{\omega}\cos\left[\omega\sqrt{3}\,(t - t_{c2})\right] \end{aligned} \qquad [3.61]$$

$$x_3^L = -x_1^R$$

As we expected, a third collision is predicted by these positions.

$$t_{c3} = 2.590037285\frac{1}{\omega} + t_{c2} \qquad [3.62]$$

[4]This value comes courtesy of Maple® V.

The leading particles again return to the origin, and they collide one last time. Afterwards, the center of mass velocities of the two clumps are

$$X^L = -0.9229314295v = -X^R \quad [3.63]$$

So the coefficient of restitution for the collision of two three-particle springy clumps is

$$e = 0.9229314295 \quad [3.64]$$

This is nearly elastic, so agreement with the cylinder collision formula is still feasible.

Of course, there are three other equal length collisions to consider. For brevity, I will present the results in a table rather than working through each collision. The analysis consists of essentially the same elements as the three-particle collision.

N	2	3	4	5	6
# of collisions	2	3	3	4	5
Duration (Units of $1/\omega$)	$\pi/\sqrt{2}$	4.752420169	4.497492158	6.823644233	9.132351359
e	1	0.9229314295	0.6599517821	0.7790646719	0.8546448130

[3.19] -- Results from the collisions of equal length clumps.

The first thing to notice is that the collisions are not elastic. The reason that the larger clumps collide even less elastically than the three-particle clump is that they collide one less time than expected. If the particles-on-strings model were any indication, we would guess that they should collide N times. Instead they collide N-1 times.

The collision of springy clumps is an interesting subject in its own right, but apparently it is not helpful in our pursuit of the behavior of cylinders and spheres. It seems that springy clumps are fundamentally different from cylinders in their behavior during collisions.

It could be suggested that we should only consider the clumps consisting of smaller numbers of particles because they are more similar to cylinders. Unfortunately, these smaller clumps lack the complexity needed to model a sphere because they do not have enough masses and springs to recreate any sort of spherical mass distribution or shape. Larger clumps are also potentially useful, as the coefficient of restitution seems to be increasing with N, but larger clumps require more powerful analysis and more computing time, and so were not feasible for this thesis.

In conclusion, models made of particles connected by Hooke's law springs behave in an interesting way during collisions. The fundamental oscillations of these objects are most easily described in an eigenspace of independent oscillations, each with its own amplitude, phase, and frequency. When a springy clump is struck, these normal modes are excited in various amounts depending on their status and the kind of impact that is

Conclusion: THE END

This thesis has explored two basic models, the particles-on-strings model and the particles-on-springs model. These models were devised in an attempt to explain the behavior of cylinders and spheres in collisions as presented by Auerbach.[1] The particles-on-strings model was successful in modeling the behavior of cylinders described by Auerbach, and the particles-on-springs model was also somewhat successful, but neither model could adequately capture the behavior of spheres during collision. These models are peculiar, however, and even though they did not produce the results we were after they are charming by themselves.

The particles on strings are the simplest examples of objects with internal degrees of freedom. They are enlightening in this aspect because they can absorb no energy during a collision or almost all the energy of a collision equally well. If Newton had considered this model when he categorized collisions, we might have a very different form of collision theory. The particles on strings "live in the details" as, with few exceptions, they require an in-depth analysis of their internal kinematics to arrive at their macroscopic behavior. They are a prime example of a system that escapes the traditional before and after treatment of collisions.

The particles on springs are also fascinating in their simultaneous simplicity and complexity. They are formed from a Lagrangian with the simplest possible interaction term, yet their behavior is sophisticated and varied. Like quantum systems, they exist as a linear combination of normal modes. These normal modes are the fundamental oscillations of the clumps. Even though a three-particle clump is a relatively simple configuration, it can exhibit three distinct motions even while limited to one dimension. The particles-on-springs model makes it clear why the behavior of molecules is a subject requiring its own discipline.

The problem of the cylinders and spheres remains unsolved, however. There are a number of directions that might be explored in finding a good explanation of this problem.

Lord Rayleigh published a paper in which he compared approximations of the spherical harmonics to the duration of impact of spheres to conclude that spheres of equal radii collide elastically as long as the speed of approach is small as compared to the speed

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[1]D. Auerbach, "Colliding Rods: Dynamics and Relevance to Colliding Balls," American Journal of Physics **62** (6) (June 1994): 522-525.

of sound in the material.[2] Rayleigh draws upon the earlier work of Hertz, who proposed a force law for spheres.[3] Love also discusses aspects of the collisions of spheres.[4]

There are also computer simulation programs that can perform sophisticated models that are distant cousins of the particles-on-springs model. These programs have been used, among other things, to simulate the impact of cars during accidents and to design explosives. One of these programs, DYNA, by John Hallquist, is available from the Lawrence Livermore National Laboratory.

The elasticity of the collision of unequal spheres as opposed to the inelasticity of the collision of unequal cylinders is still mystifying. Perhaps a new approach or more sophisticated use of these models will reveal the cause of this behavior.

---

[2]Lord Rayleigh, "On the Production of Vibrations by Forces of Relatively Long Duration, with Application to the Theory of Collision," Phil. Mag. **11** (1906): 283.

[3]Hertz's work in collision theory is reproduced in English in Love, A.E.H., *A Treatise on the Mathematical Theory of Elasticity*. 4th ed. (New York: Dover Publications 1944) pp.198-200.

[4]*Ibid.,* pp. 199-201.

Appendix A:
The Balls on Strings Program

```
! Balls on strings Dynamics Ver. 1 - Baylor Fox
!This program will simulate the interactions of two non-rigid bodies
!made of one-dimensional chains of particle masses connected with massless
!strings.  It will also keep track of the centers of mass
!and the elasticity of the collision.  It reads input data from a separate file.

SET COLOR "black"
PRINT "WELCOME TO THE BALLS ON STRINGS SIMULATOR!"
INPUT prompt "ready":h$

LET rmax=10000                !Maximum # of iterations

DIM pos(1 to 50,0 to 1), vel(1 to 50,0 to 1)     !Declare Necessary Arrays
DIM str(1 to 50), mass(1 to 50), M(1 to 50), T(0 to 10001)
DIM CM(1 to 4,1 to 50,0 to 1), twang$(1 to 50), pn(1 to 50)
DIM Energy(0 to 1), CMenergy(0 to 1)

LET Numero=0
OPEN #1:name "Data"           !Open the initial conditions file

DO
  LET Numero=Numero+1

  !zero the positions and velocities
  MAT pos=zer
  MAT vel=zer
  MAT str=zer
  MAT pn=con
  LET T(0)=0
  LET r=0

  CALL readdata            !Read initial conditions
  CALL findparts           !Find where the break in the chains is
  CALL partmass             !Calculate the masses of the two chains

  FOR i=1 to n
    LET pos(i,1)=pos(i,0)
    LET vel(i,1)=vel(i,0)
  NEXT i

  CALL openfiles            !Open the output files

  CALL pstats              !Calculate Information about the centers of mass
  CALL energies             !Calculate the energies
  CALL sendata              !Send the initial data to output files

  FOR i=1 to pn(n)
    LET CM(1,i,0)=CM(1,i,1)
    LET CM(2,i,0)=CM(2,i,1)
    LET CM(3,i,0)=CM(3,i,1)
    LET CM(4,i,0)=CM(4,i,1)
  NEXT i
```

```
LET Energy(0)=Energy(1)
LET CMEnergy(0)=CMEnergy(1)


! The Simulation Begins!
DO
  IF r/250=int(r/250) then PRINT  r;"...";Numero;"...";  CM(1,1,1); "..."; CM(1,2,1)
  CALL pstats
  CALL energies
  CALL sendata

  !Find the next collision time
  LET t0=20000
  FOR i=1 to n-1
     IF vel(i+1,1)-vel(i,1)<0 then
        CALL CALCTIME(pos(i,1),vel(i,1),pos(i+1,1),vel(i+1,1),T(r),t0)
     ELSE IF vel(i+1,1)-vel(i,1)>0 and str(i)>0 then
        CALL CALCTIME(pos(i+1,1)-str(i),vel(i+1,1),pos(i,1),vel(i,1),T(r),t0)
     END IF
  NEXT i

  !Find next particle leaving collision zone time
  FOR i=1 to n
     IF vel(i,1)<>0 and pos(i,1)<5000 and pos(i,1)>0 then
        CALL LeaveTIME(pos(i,1),vel(i,1),T(r),t0)
     END IF
  NEXT i

  LET T(r+1)=t0+.001          !set the next time

  FOR i=1 to n               !move the particles
     CALL MOVE(pos(i,1), vel(i,1), T(r), T(r+1))
  NEXT i

  LET r=r+1                  !set forward the iteration #

  !collide the coincident particles
  CALL collider


  LET q=0                    !decide when to stop the simulation
  IF t0=>10000 then
     LET q=1
     PRINT "out of time"
  END IF
  IF r>=rmax then
     LET q=1
     PRINT"You've hit the MAX!"
  END IF
  FOR i=1 to n
     IF pos(i,1)>=5000 then
        LET q=q+1/n
        IF pos(i,1)<=0 then
           LET q=q+1/n
        END IF
        IF q>0.8 then PRINT"Out of Bounds!"
     END IF
  NEXT i
```

```
  LOOP until q>=.98

  CALL pstats
  CALL energies
  CALL sendata
  CALL Finish

  CLOSE #2
LOOP
CLOSE #1
CLOSE #3
SOUND 440,15

!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!subroutines!!!!!!!!!!!!!!!!!!!!!!!
SUB readdata
   LINE INPUT #1: Crapola$
   INPUT #1: n                    !Get number of masses
   IF n=42 then
     SOUND 440,10
     STOP
   END IF                      !Deciding to stop
   LINE INPUT #1: Crap$
   FOR i=1 to n                !Get Masses
      INPUT #1: mass(i)
   NEXT i
   LINE INPUT #1: Crap$
   FOR i=1 to n                !Get String Lengths
      INPUT #1: str(i)
   NEXT i
   LET str(n)=0
   LINE INPUT #1: Crap$
   FOR i=1 to n                !Get initial positions
      INPUT #1: pos(i,0)
   NEXT i
   LINE INPUT #1: Crap$
   FOR i=1 to n                ! Get the initial velocities
      INPUT #1: vel(i,0)
   NEXT i
END SUB

SUB collider
   FOR i=1 to n-1
      IF pos(i+1,1)<=pos(i,1) and vel(i+1,1)<vel(i,1) then    !collide
        LET twang$(i)="BANG!"
      ELSE IF pos(i+1,1)-str(i)=>pos(i,1) and str(i)>0 and vel(i+1,1)>vel(i,1) then    !twang
        LET twang$(i)="TWANG!"
      ELSE
        LET twang$(i)=" "
      END IF
   NEXT i

  LET decider=0
   FOR i=1 to n-2              !How many collide?
     IF twang$(i)<>" " and twang$(i+1)<>" " then LET decider=1
   NEXT i
   IF decider>=1 then          !If all num+1 collide, randomly change
     RANDOMIZE                     !All particles initial position, following
    PRINT "adjusting for indeterminate collision"
```

```
    FOR i=1 to n
      DO                  !constraints and restart the entire process
        LET pos(i,0)=pos(i,0)+RND-0.5
        LET quiz=0
        IF i>1 then       !Check constraints on the left
          IF pos(i,0)<pos(i-1,0) then LET quiz=1
          IF str(i-1)>0 and pos(i,0)>pos(i-1,0)+str(i-1) then LET quiz=1
        END IF
        IF i<n then       !CHeck constraints on the right
          IF pos(i,0)>pos(i+1,0) then LET quiz=1
          IF str(i)>0 and pos(i,0)+str(i)<pos(i+1,0) then LET quiz=1
        END IF
      LOOP until quiz=0
    NEXT i
    LET r=0
    FOR i=1 to n
      LET pos(i,1)=pos(i,0)
      LET vel(i,1)=vel(i,0)
    NEXT i
    CALL RESETER
    EXIT SUB                !Restart with new positions
  END IF

  FOR i=1 to n-1            !perform the remaining two-body collisions
    IF twang$(i)="BANG!" or twang$(i)="TWANG!" then    !collide
      CALL COLLIDE(mass(i),vel(i,1),mass(i+1),vel(i+1,1))
    END IF
  NEXT i
END SUB

SUB sendata                 !Save Data of Each Collision
  !Send Position and time updates to the plotting files
  RESET #2:end
  PRINT #2: T(r);
  FOR i=1 to n
    PRINT #2: pos(i,1);
  NEXT i
  PRINT #2: ""
END SUB

SUB findparts               !find # of Clumps
  LET c=1
  FOR i=1 to n-1
    IF str(i)=0 then
      LET pn(i)=c
      LET c=c+1
    ELSE
      LET pn(i)=c
    END IF
  NEXT i
  LET pn(n)=c
  LET pn(n+1)=c+1
END SUB

SUB partmass                !find mass of clumps
  LET j=1
  FOR i=1 to pn(n)
    LET M(i)=0
```

```
    DO
      LET M(i)=M(i)+mass(j)
      LET j=j+1
    LOOP until pn(j)>i
  NEXT i
END SUB

SUB pstats                    !find CM of clumps pos, vel, energies
  LET j=1
  FOR i=1 to pn(n)
    LET CM(1,i,1)=0
    LET CM(2,i,1)=0
    LET CM(3,i,1)=0
    DO
      LET CM(1,i,1)=CM(1,i,1)+mass(j)*pos(j,1)/M(i)
      LET CM(2,i,1)=CM(2,i,1)+mass(j)*vel(j,1)/M(i)
      LET CM(3,i,1)=CM(3,i,1)+0.5*mass(j)*((vel(j,1))^2)
      LET j=j+1
    LOOP until pn(j)>i
    LET CM(4,i,1)=0.5*M(i)*((CM(2,i,1))^2)
  NEXT i
END SUB

SUB energies                  !Find energy values
  LET CMenergy(1)=0
  LET Energy(1)=0
  FOR i=1 to pn(n)
    LET Energy(1)=Energy(1)+CM(3,i,1)
    LET CMenergy(1)=CMenergy(1)+CM(4,i,1)
  NEXT i
END SUB

SUB Finish                    !Prints the final energy measurements
  PRINT "Initial Energy:";Energy(0)
  PRINT "Final Energy:";Energy(1)
  PRINT "Initial Energy From CM Motion:";CMenergy(0)
  PRINT "Final Energy From CM Motion:";CMenergy(1)
  FOR i=1 to pn(n)-1
    PRINT "Cffnt of Restitution between clump#";i;" and clump#";
    PRINT i+1;":";abs(CM(2,i,1)-CM(2,i+1,1))/abs(CM(2,i,0)-CM(2,i+1,0))
  NEXT i
  PRINT crapola$
  PRINT #3:Crapola$
  PRINT #3:"Initial Energy:";Energy(0)
  PRINT #3:"Final Energy:";Energy(1)
  PRINT #3:"Initial Energy From CM Motion:";CMenergy(0)
  PRINT #3:"Final Energy From CM Motion:";CMenergy(1)
  FOR i=1 to pn(n)-1
    PRINT #3: "Cffnt of Restitution between clump#";i;" and clump#";
    PRINT #3:i+1;":";abs(CM(2,i,1)-CM(2,i+1,1))/abs(CM(2,i,0)-CM(2,i+1,0))
  NEXT i

END SUB

SUB openfiles
  OPEN #2: name "Datapts"&Str$(Numero/1000),create newold
  ERASE #2
  SET #2:MARGIN 10^100
```

```
  OPEN #3:name "Final",create newold
   ERASE #3
END SUB

SUB RESETER
   ERASE #2
END SUB

END

SUB LeaveTIME(x,v,t,t0)
  LET tL=max((5001-x)/v+t,(-1-x)/v+t)
   IF tL<t0 and tL>t+0.001 then LET t0=tL
END SUB

SUB CALCTIME(x, v, xi, vi, t, t0)      ! Sub to figure out collision time
   IF x=xi then
     LET ti=10^10
   ELSE
     LET ti=(xi-x)/(v-vi)+t
   END IF
   IF ti<t0 and ti>t+0.001 then LET t0=ti
END SUB

SUB MOVE(x0, v, t0, tf)          ! Sub to move a particle
  LET xf=x0+v*(tf-t0)
  LET x0=xf
END SUB

SUB collide (m,v,mi,vi)
  LET u=((m-mi)*v+2*mi*vi)/(m+mi)
  LET ui=(2*m*v+(mi-m)*vi)/(m+mi)
  LET v=u
  LET vi=ui
END SUB
```

Appendix B:
An Example Input File

```
Cylinder1 5 5
 10
Mass Array
 1
 1
 1
 1
 1
 1
 1
 1
 1
 1
String Array
 62.5
 62.5
 62.5
 62.5
 0
 62.5
 62.5
 62.5
 62.5
 0
position array
-200
-150
-100
-50
 0
 5000
 5050
 5100
 5150
 5200
Velocity Array
 1
 1
 1
 1
 1
-1
-1
-1
-1
-1
Cylinder1 5 7
 12
Mass Array
 1
 1
 1
 1
```

1
1
1
1
1
1
1
1
String Array
62.5
62.5
62.5
62.5
0
58.3333
58.3333
58.3333
58.3333
58.3333
58.3333
0
position array
-200
-150
-100
-50
0
5000
5050
5100
5150
5200
5250
5300
Velocity Array
1
1
1
1
1
-1
-1
-1
-1
-1
-1
-1
END
42

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